MULTIPLICATION OPERATIONS ON VECTORS AND SOME OF THEIR APPLICATIONS

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Multiplication operations in the forms of scalar and vector products have been dealt with in NCERT textbook for Class XII. In this article, we describe further extension of products and discuss some examples related to some physical situations and some specific problems based on these concepts. We close this article by giving some remarks on crucial concepts of projection and distance between skew lines in vector forms.

1. Introduction

In the case of vectors, we find that two multiplication operations are defined for a pair of vectors. One operation called a scalar product generates a scalar, whereas the other operation called a vector product generates a vector.

Expressions of the form $\vec{a}\cdot\vec{b}\cdot\vec{c}$, $\vec{a}\,.\vec{b}\;\cdot\vec{c}\;\cdot\vec{d}$ are meaningless since the scalar product is only defined between a pair of vectors. The operation of division of vectors is not defined. We may write $\vec{a}\cdot\vec{b} = c$, but it is absurd to write $\vec{a} = \frac{c}{\vec{b}}$ or $\vec{b} = \frac{c}{\vec{a}}$. Another form of product of two vectors is the vector product. We denote the vector products of $\vec{a}$ and $\vec{b}$ by $\vec{a}\times\vec{b}$. With the help of scalar product and vector product, the mixed product of the form $\vec{a}\cdot(\vec{b}\times\vec{c})$ become possible. The next generalisation is the vector product of more than two vectors defined in specified way as $\vec{a}\times(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\times\vec{c}$.

For completeness, we elaborate upon scalar triple product and product of more than two vectors as these have not been discussed in NCERT textbook.

1.1 Scalar Triple Product

Let
$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

Then,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Reader may verify that, using the properties of determinants, we obtain

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

or,

$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$

Geometrically, $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ gives the volume of a parallelopiped whose coterminous edges are represented by vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively (see Fig.1)

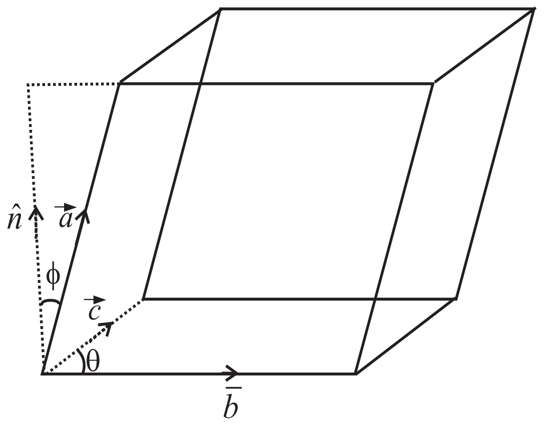

Fig. 1

The area of the parallelogram OBCD is $|\vec{b} \times \vec{c}|$ and the direction of $\vec{b} \times \vec{c}$ is along $\hat{n}$ which is normal to the plane containing $\vec{b}$ and $\vec{c}$. Let *h* denote the height of the terminal point of vector $\vec{a}$ above the parallelogram OBCD. The volume of the parallelepiped

$= h|\vec{b} \times \vec{c}|$ (height × area of parallelogram OBCD)

$= |\vec{a} \cdot \hat{n}||\vec{b} \times \vec{c}|$

$= |\vec{a} \cdot |\vec{b} \times \vec{c}|\hat{n}|$

$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$

1.2 Vector Product of More than Two Vectors

Let,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Using basic definition of vector product of vectors and its representation in determinant form, one can readily deduce that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Geometrically, $\vec{b} \times \vec{c}$ is perpendicular to the plane containing $\vec{b}$ and $\vec{c}$, so $\vec{a} \times (\vec{b} \times \vec{c})$ lies in the plane containing $\vec{b}$ and $\vec{c}$.

2. Application in Physical Situations

2.1. Work Done by Force

If force $\vec{F}$ acts on a body causing displacement $\vec{d}$ in the direction of force $\vec{F}$, then the work done by the force $\vec{F}$ is defined as the product of the distance moved and the component of force in the direction of the displacement. Hence,

W (workdone) $= |\vec{F}| \cos\theta |\vec{d}|$

$= |\vec{F}||\vec{d}| \cos\theta$

$= \vec{F} \cdot \vec{d}$

As W is always positive, so

$$W = |\vec{F} \cdot \vec{d}|$$

Example 1: Let the force $\vec{F}_1$ of magnitude 5 units be in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$, force $\vec{F}_2$ of magnitude 4 units in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$ and force $\vec{F}_3$ of magnitude 3 units in the direction of $-2\hat{i} + \hat{j} - 2\hat{k}$ act on a particle A (6, 2, 3) displacing it to the new position B (9, 7, 5). Find the work done.

Solution: Here

$$\vec{F}_1 = \frac{5}{3}(2\hat{i} - 2\hat{j} + \hat{k})$$

$$\vec{F}_2 = \frac{4}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$\vec{F}_3 = \frac{3}{3}(-2\hat{i} + \hat{j} - 2\hat{k})$$

So, the total force acting on the particle A is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= \frac{1}{3}[8\hat{i} + \hat{j} + 7\hat{k}]$$

Taking **O** as fixed point, we can assign position vector to the point A as $\overrightarrow{OA} = 6\hat{i} + 2\hat{j} + 3\hat{k}$ and position vector of the point **B** as $\overrightarrow{OB} = 9\hat{i} + 7\hat{j} + 5\hat{k}$.

Therefore, displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= 3\hat{i} + 5\hat{j} + 2\hat{k}$$

Hence work done $W = \vec{F} \cdot \vec{d}$

$$= \frac{1}{3}(8\hat{i} + \hat{j} + 7\hat{k}) \cdot (3\hat{i} + 5\hat{j} + 2\hat{k})$$

$$= \frac{1}{3}[24 + 5 + 14]$$

$$= \frac{43}{3} \text{ units}$$

2.2. Moment (torque) of the Force and the Moment of Momentum

If the force $\vec{F}$ acts on a point P of the body through the point whose position vector is $\vec{r}$ with respect to the specified point O as shown in the Figure 2, then the moment of the force $\vec{F}$ about the point O is defined as

$$\vec{M} = \vec{r} \times \vec{F}$$

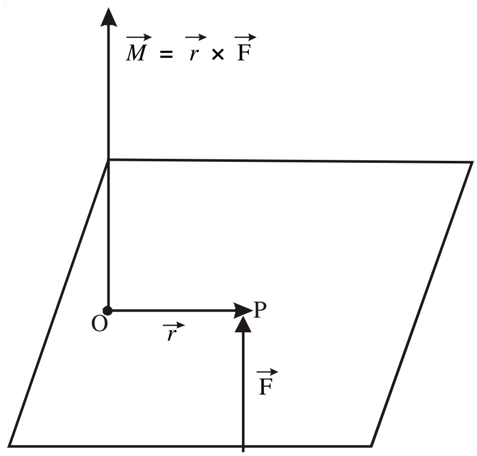

Fig. 2

If $\vec{F}$ is identified with momentum $m\vec{V}$ of the particle of mass *m* at P moving with velocity $\vec{V}$ then the moment of momentum about the fixed point O is expressed as:

$$\vec{M} = \vec{r} \times m\vec{V}$$

$$= m\vec{r} \times \vec{V}$$

It is also called the angular momentum of the particle about O.

Example 2: Find the torque of a force represented by $3\hat{i} + 6\hat{j} + \hat{k}$ about the point O given that it acts through the point A (–1, 1, 2) relative to the point O.

Solution: Here, $\vec{F} = 3\hat{i} + 6\hat{j} + \hat{k}$ and the position vector of the point A with respect to O can be expressed as $\vec{r} = -\hat{i} + \hat{j} + 2\hat{k}$. Therefore, torque of the force

$$\vec{M} = \vec{r} \times \vec{F}$$

$$= (-\hat{i} + \hat{j} + 2\hat{k}) \times (3\hat{i} + 6\hat{j} + \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 3 & 6 & 1 \end{vmatrix}$$

$$= \hat{i}(1-12) - \hat{j}(-1-6) + \hat{k}(-6-3)$$

$$= -11\hat{i} + 7\hat{j} - 9\hat{k}$$

2.3 Angular Velocity

Consider a point P on the rigid body rotating about the axis L through the point O with constant spin ω (angular speed).

Let $\vec{d}$ represent the position vector of the point P with respect to O. Since P travels in a circle of radius $d\sin\theta$ (see Fig.3) having centre at C, the instaneous linear velocity $\vec{v}$ along the tangent at P has its magnitude,

$$\omega\,(d\sin\theta) = |\,\vec{\omega} \times \vec{d}\,|$$

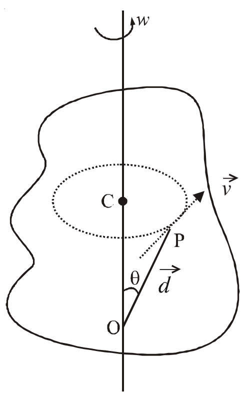

Fig. 3

Also $\vec{v}$ must be perpendicular to both $\vec{\omega}$ and $\vec{d}$ in such a way that $\vec{d}, \vec{\omega}$ and $\vec{v}$ form a right-handed system.

Example 3: A body spins about a line through the origin parallel to the vector $2\hat{i} - \hat{j} + \hat{k}$ at 15 rad/s. Find the angular velocity of the body and also instantaneous linear velocity of a point in the body with position vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

Solution: Angular velocity

$$\vec{\omega} = |\vec{\omega}| \frac{(2\hat{i} - \hat{j} + \hat{k})}{\sqrt{6}}$$

$$= \frac{15}{\sqrt{6}}(2\hat{i} - \hat{j} + \hat{k})$$

Instantaneous linear velocity

$$\vec{v} = \vec{\omega} \times \vec{d}$$

Here, $\vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Therefore, $\vec{v} = \vec{\omega} \times \vec{d}$

$$= \frac{15}{\sqrt{6}}(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= \frac{15}{\sqrt{6}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \frac{15}{\sqrt{6}}[\hat{i}(-3-2) - \hat{j}(6-1) + \hat{k}(4+1)]$$

$$= \frac{75}{\sqrt{6}}(-\hat{i} - \hat{j} + \hat{k}).$$

3. Some Specific Problems Based on Product of Vectors

Problem 1: From a given point and a given line, how will you determine the perpendicular distance of the point from the line?.

Problem solving: Let C be the point with position vector $\vec{c}$ and A be the point on the line L with position vector $\vec{a}$ with reference to the fixed point O (see Fig.4). Assume that given line L is parallel to the vector $\vec{b}$, so the equation of the line L can be expressed as

$\vec{r} = \vec{a} + \lambda\vec{b}$ where l is scalar and $\vec{r}$ is the position vector of any arbitrary point P (say) on the line L.

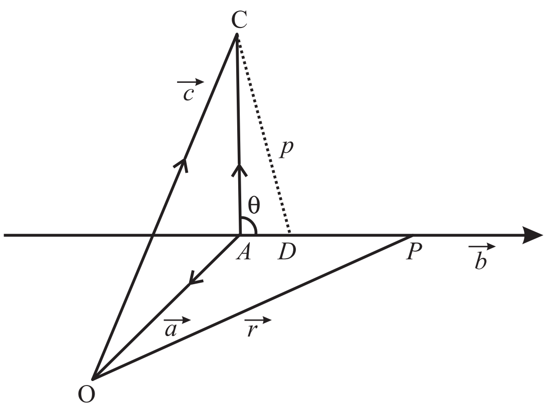

Fig. 4

From Fig.4, p is the perpendicular distance from the given point C to the given line which has to be determined. From the Fig.

$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \vec{c} - \vec{a}$$

From the triangle ACD, we have

$$p^2 = (AC)^2 - (AD)^2$$

$$= \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AD}\right|^2 \qquad (1)$$

Obviously, AD is the projection of AC on the line L, i.e.,

$$\left|\overrightarrow{AD}\right| = AD = AC\cos\theta$$

$$= \overrightarrow{AC}\cdot\frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{(\vec{c}-\vec{a})\cdot\vec{b}}{|\vec{b}|}$$

Hence from (1), we have

$$p^2 = (\vec{c}-\vec{a})^2 - \left(\frac{(\vec{c}-\vec{a})\cdot\vec{b}}{|\vec{b}|}\right)^2.$$

Thus, p can be deduced from the above expression.

Problem 2: Consider a tetrahedron with faces F_1, F_2, F_3 and F_4. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ be vectors whose magnitudes are respectively equal to the areas of F_1, F_2, F_3 and F_4 and whose directions are perpendicular to these faces in the outward direction. Then how to determine $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4$?

Problem Solving

Consider the following Fig. 5 representing the tetrahedron OABC with faces F_1, F_2, F_3 and F_4. Let $\vec{a}, \vec{b}, \vec{c}$ represent the position vectors of the vertices A, B and C, respectively.

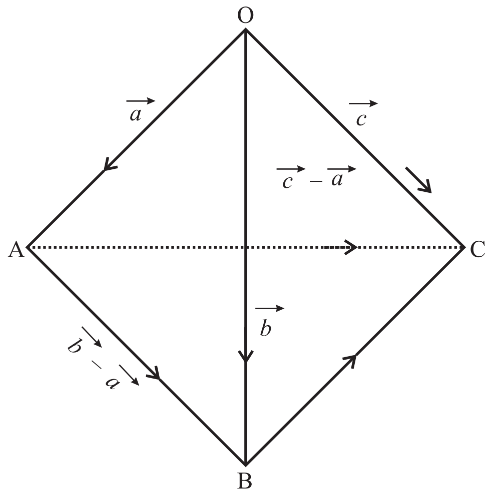

Fig. 5

Again, area of

$$F_1 = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$$

area of $F_2 = \frac{1}{2}\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{1}{2}\left|\vec{b} \times \vec{c}\right|$

area of $F_3 = \frac{1}{2}\left|\overrightarrow{OC} \times \overrightarrow{OA}\right| = \frac{1}{2}\left|\vec{c} \times \vec{a}\right|$

area of $F_4 = \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{AB}\right| = \frac{1}{2}\left|(\vec{c} - \vec{a}) \times (\vec{b} - \vec{a})\right|$

According to the problem,

$$\vec{v}_1 = \frac{1}{2}(\vec{a} \times \vec{b})$$

$$\vec{v}_2 = \frac{1}{2}(\vec{b} \times \vec{c})$$

$$\vec{v}_3 = \frac{1}{2}(\vec{c} \times \vec{a})$$

and $\vec{v}_4 = \frac{1}{2}\left\{(\vec{c} - \vec{a}) \times (\vec{b} - \vec{a})\right\}$

Again, $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 =$

$$\frac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b} - \vec{c} \times \vec{a} - \vec{a} \times \vec{b}\right]$$

$$= .\ \vec{0}$$

Problem 3: If $\vec{a} = 2\hat{i} + \hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ and the vector $\vec{r}$ satisfies the relationship $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0,$ then how to find $\vec{r}$?

Problem Solving: Given that,

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

$$\Rightarrow \quad \vec{a} \times (\vec{r} \times \vec{b}) = \vec{a} \times (\vec{c} \times \vec{b})$$

$$= (\vec{a} \cdot \vec{b})\vec{r} - (\vec{a} \cdot \vec{r})\vec{b}$$

$$= (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$$

$$= (\vec{a} \cdot \vec{b})\vec{r} = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$$

(Since $\vec{r} \cdot \vec{a} = 0$)

$$= 3\vec{r} = 3\vec{c} - 15\vec{b}$$

$$= \vec{r} = \vec{c} - 5\vec{b}$$

$$= -\hat{i} - 8\hat{j} + 2\hat{k}.$$

Problem 4: Let $\vec{b} = 4\hat{i} + 3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in *xy*-plane. How to determine a vector in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively?

Problem solving: Let $\vec{r}$ represent the vector to be determined in the *xy*-plane containing $\vec{b}$ and $\vec{c}$ respectively, i.e., $\vec{r}$ is coplanar with $\vec{b}$ and $\vec{c}$ and hence $\vec{r}$ can be expressed as

$\vec{r} = \lambda\vec{b} + \mu\vec{c}$ for l, m being some scalar.

$\vec{c}$ is the given vector in the same plane so, we can write $\vec{c} = x\hat{i} + y\hat{j}$ for some scalars *x* and *y*.

Since, $\vec{c}$ and $\vec{b}$ are perpendicular to each other so $\vec{c}\cdot\vec{b} = 0 \Rightarrow 4x + 3y = 0$ which gives $y = -\frac{4}{3}x$.

or $\vec{c} = x\left(\hat{i} - \frac{4}{3}\hat{j}\right)$

Given that projection of $\vec{r}$ along $\vec{b}$ is 1, i.e.,

$$\vec{r}\cdot\frac{\vec{b}}{|\vec{b}|} = 1$$

$$\Rightarrow \qquad \frac{(\lambda\vec{b} + \mu\vec{c})\cdot\vec{b}}{|\vec{b}|} = 1$$

$$\Rightarrow \qquad 51 = 1$$

or $$1 = \frac{1}{5}$$

Similarly, projection of $\vec{r}$ on $\vec{c}$ is 2, i.e.,

$$\vec{r}\cdot\frac{\vec{c}}{|\vec{c}|} = 2 \Rightarrow \mu|\vec{c}| = 2$$

or $$\mu x = \frac{6}{5}$$

Hence $$\vec{r} = \frac{1}{5}(4\hat{i} + 3\hat{j}) + \frac{6}{5}(\hat{i} - \frac{4}{3}\hat{j})$$

$$= 2\hat{i} - \hat{j}.$$

4. Remark's on Projection and Distance between Skew-lines

In NCERT textbook, Class XII, Part II, basic concept on projection of a vector on a line has been explained on page 443 and its important application in finding the distance between the skew lines in space has also been discussed (see page 474). These concepts are crucial and are used in many physical situations besides being used in the study of vectors and three dimensional geometry. Therefore, giving more deeper insight into these concepts as remark's will supplement the textual material on these concepts given in above cited NCERT textbook.

4.1. Projection of a Vector on a Line

Let L be a given line and AB a given directed line segment in three-dimensional space. Intuitively, to make the definition of projection of AB on L, we erect planes passing through points A and B perpendicular to L (see Fig.6). These planes cut the line L at A' and B', respectively. Thus, we obtain the directed line segment A'B' which is called projection of AB onto the line L.

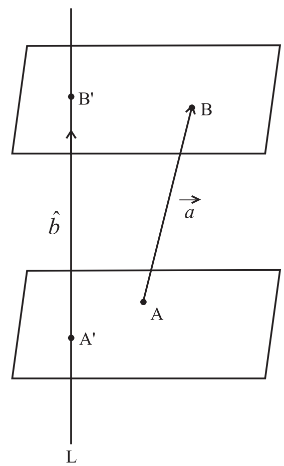

Fig. 6

In Literature, generally projection of a vector on a line is taken in both senses either as a scalar or as a vector. However, projection as a vector is more useful in physical sciences as well as in engineering. Let us distinguish between these two projections.

Consider $\overrightarrow{AB} = \vec{a}$ and suppose $\hat{b}$ is along the line L, then the projection (as vector) of $\vec{a}$ on the line L is given by $\left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}\right) \frac{\vec{b}}{|\vec{b}|}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right) \vec{b}.$$

Projection (as scalar) of $\vec{a}$ on L is simply expressed as

$$\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}.$$

It is interesting to exhibit another way of looking at the projection. To this end, let $\vec{a}$ be the given vector whose projection on the line L is to be determined.

Let $\vec{b}$ be the vector along the line L determining its direction and $\vec{p}$ denote the projection vector of $\vec{a}$ on the line L (see Fig.7).

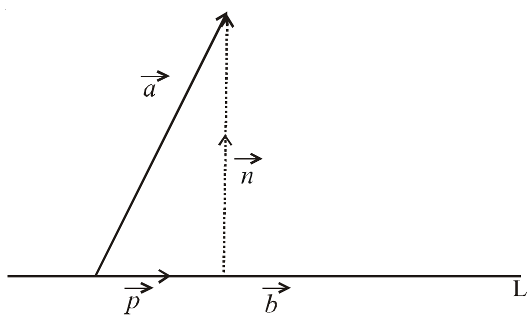

Fig. 7

Clearly, $\vec{n} = \vec{a} - \vec{p}$.

Since $\vec{p}$ and $\vec{b}$ are collinear vectors, so

$\vec{p} = \lambda\vec{b}$ (l is a scalar).

Again $\vec{n}$ is perpendicular to $\vec{b}$, we must have

$$\vec{n} \cdot \vec{b} = \vec{a} \cdot \vec{b} - \lambda\vec{b} \cdot \vec{b}$$

$$0 = \vec{a} \cdot \vec{b} - \lambda|\vec{b}|^2$$

or $$l = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}.$$

Hence, projection vector $\vec{p} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right) \frac{\vec{b}}{|\vec{b}|}$.

Now, we wish to turn to the problem of finding a formula for the distance between the two skew lines. The distance formula has been derived in NCERT textbook, Class XII, Part II on page 474, however, the same is being discussed in more intuitive way so as to strengthen the understanding of shortest distance between the two skew lines in space. Let the line L_1 parallel to the vector $\vec{b}_1$ passing through the point A_1 with position vector $\vec{a}_1$ and the other line L_2 parallel to the vector $\vec{b}_2$ and passing through the point A_2 with position vector $\vec{a}_2$. As these lines are in space, we can think of two different planes containing lines L_1 and L_2, respectively, as shown in Fig. 8.

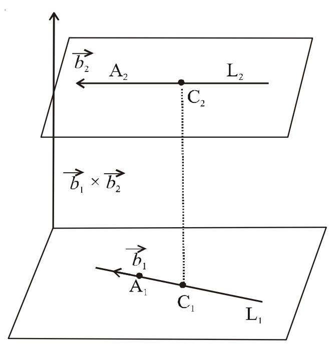

Fig. 8

We can find point C_1 on L_1 and point C_2 on L_2 such that C_1C_2 is perpendicular to both lines. In fact C_1C_2 is the shortest distance between lines L_1 and L_2 which happens to be the distance between the two planes containing lines L_1 and L_2, respectively.

$\vec{b}_1 \times \vec{b}_2$ is also perpendicular to these two planes, i.e., $\overrightarrow{C_1C_2}$ and $\vec{b}_1 \times \vec{b}_2$ are collinear. Thus, we can think of C_1C_2 as the projection of $\overrightarrow{A_1A_2}$ in the direction of $\vec{b}_1 \times \vec{b}_2$ and hence, shortest distance $\left|\overrightarrow{C_1C_2}\right|$ between lines L_1 and L_2 is

$$\left|\frac{\overrightarrow{A_1A_2} \cdot \vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$= \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}.$$

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