Why is it Difficult to Understand Fractions?

Vyomesh Pant*

Abstract

Fractions, decimals and percentage are among such topics which are taught in the schools during the initial years of learning mathematics. However, due to poor conceptualisation, the students often find it difficult to understand these topics and a perception starts to grow among the children that mathematics is a difficult subject to study and everyone cannot learn it.

The present paper is intended to find out the reasons which make it difficult to understand fractions, be it the inherent nature of fractions or the methods generally used to teach fractions or the psychological reasons related to cognition of the children. The paper discusses the reasons behind difficulties in teaching and learning of fractions, which may give an insight to the teachers about these difficulties from the point of view of the beginners so that they can prepare accordingly to deliver lessons on fractions.

Introduction

Fractions play a key role in the learning of mathematics. These are widely used not only in mathematics but also in economics, engineering, physics, chemistry, biology, medical science and other academic offshoots. Fractions are among the topics taught in school during the initial years of learning mathematics. Nevertheless, due to non-clarity of conceptions, students often wrongly perceive mathematics to be a difficult subject. Infact, the students, for a variety of reasons, are not able to understand fractions properly. As a result, due to poor conceptualisation, children cannot solve the basic problems related to fractions. For example, on a national test, only 50 per cent of American

**Independent Researcher*, Ministry of Finance, Department of Economic Affairs. Govt. of India, New Delhi 110 001; vyomeshpant@gmail.com

eighth graders correctly ordered three fractions from the smallest to the largest (see National Council of Teachers of Mathematics, 2007). Even in countries where the majority of students do achieve reasonably good conceptual understanding, such as Japan and China, fractions are considered a difficult topic (Fazio and Siegler, 2011).

Fractions are widely recognised as a very important topic in mathematics which is difficult to teach for the teachers and difficult to understand for the children as well. Teaching of fractions poses a significant challenge for the teachers to present it in such a way that can be adapted easily by the children. It has been found that the understanding and conceptualisation of fractions is weak among children of all age groups around the globe, which creates difficulty for them in learning fractions and they often remain confused (Armstrong and Larson, 1995; Behr, Lesh, Post and Silver, 1983; Clarke and Roche, 2010; Condon and Hilton, 1999; Erlwanger, 1973; Empson and Levi, 2011; Kamii and Clark, 1995; Mack, 1990; Ma, 1999; Moss and Case, 1999; Orpwood, Schollen, Leek, Marinelli-Henriques and Assiri, 2012; Post, Cramer, Lesh, Harel, and Behr, 1993; Perle, Moran and Lutkus, 2005; Stigler, Givvin and Thompson, 2010). The present paper is intended to find out the factors which make the learning of difficult fractions.

Various Aspects in Teaching of Fractions Vis-à-Vis the Difficulties in Understanding and Conceptualisation

The Part-whole Concept

Fractions are normally introduced as 'part-whole' with the help of a circle, a square, a pizza or a bread, by dividing it into 2 or 4 equal parts to introduce the concept of $\frac{1}{2}$ or $\frac{1}{4}$ and so on. Following the similar practice with other geometric figures, the concept of $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, etc., is provided to the learners. In all such fractions, the numerator is initially kept as 1. Then the concept of the fractions like $\frac{2}{5}, \frac{5}{7}, \ldots$ are given to the learners in which numerator is some number other than 1. Thereafter, the concept is generalised and fractions are introduced in the form $\frac{x}{y}$, where x and y are two whole numbers, x is called the numerator and y, the denominator.

The part-whole concept, on its own, is not a sufficient foundation for conceptual understanding of fractions (Clarke et al., 2011). It does not work in case of fractions having value greater than 1. Also, it is not possible to explain the basic algebraic operations on fractions on the basis of the part-whole concept. Sometimes, $\frac{x}{y}$ is taught as 'x out of y', the students are asked to divide a geometrical shape into y number of equal parts and then to choose x number of parts out of it to understand the meaning of $\frac{x}{y}$. However, it is always not easy for the learners to divide a geometric shape into equal parts and the exercise of equal division draws a considerable amount of attention. For example, in case of a circle, it is not easy to divide it into 5 or 7 equal parts and accordingly, it is not easy to explain the fractions like $\frac{2}{5}, \frac{1}{7}, \frac{3}{7}$ using a circle. Another problem is that the method is not able to explain improper fractions like $\frac{9}{7}$, '9 parts out of 7 equal parts'.

Fractions are Generally not Introduced in a Natural Way

Unfortunately, under the procedure generally adopted by the teachers, the introduction of fraction does not come in a natural way for the beginners. For the learners, fraction becomes entirely a new aspect which is normally taught to them without associating it with their previous knowledge and without telling them the genesis behind introducing fractions. The mathematics of early years is confined to learning of numbers only. Whole numbers, natural numbers, integers, their magnitudes, position of numbers in the number line, arranging numbers in ascending and descending orders are among the basic problems which are part of mathematics during initial few years of learning. It may, therefore, be in the interest of the learning of the children if introduction of fractions is connected to the number knowledge achieved by them. However, normally it does not happen. Often the fractions are taught without making its connection with the number system. In fact, a fraction is a number; however, since it is not introduced in that manner, the learners often visualise it as a structure made up from numbers.

Fractions are Introduced as a New Aspect, Different from Numbers

The fractions are taught in such a way that they represent a new mathematical entity having three parts—numerator, denominator and a line separating them. If not taught properly, most of the learners remain more focused on the three parts and less on the fraction. They cannot conceptualise the fraction as a whole. Researchers have reported that a considerable amount of attention of the children gets utilised to understand these three entities (numerator, denominator and bar) and a less amount is left to understand the structure and operations related to fractions. The greater memory load of representing fractions reduces the cognitive resources available for thinking about the procedure needed to solve the problems related with fractions (Lortie-Forgues, et al., 2015; Fuchs, et al., 2013; 2014; Hecht and Vagi, 2012; Jordan et al., 2013; Siegler and Pyke, 2013).

Prior to learning fractions, the students are aware of the whole numbers, natural numbers, integers, etc. With their background about the number knowledge they find fractions as a mathematical entity made from two whole numbers, but they cannot understand that the fraction itself is a number, which is a major threshold in understanding fractions.

Various Forms are Used to Introduce Fractions

Kieren (see Kieren; 1980) had identified five different interpretations (or sub-constructs) of rational numbers, which are often summarised as part-whole, measure, quotient or division, operator and ratio. Fraction models or notations can be used in all of these five contexts and during teaching, all these models are taught to the children. The problem is that without conceptualisation of fractions, the children are not able to correlate these forms and remain confused. A fraction in the form of $\frac{x}{y}$ denotes a rational number, where x and y are whole numbers. Alternatively, $\frac{x}{y}$ can be said as '*x* divided by *y*' or $x \div y$, that means $\frac{x}{y}$ is the quotient when a whole number *x* is divided by another whole number *y*. $\frac{x}{y}$ also denotes the ratio of two whole numbers *x* and *y*. In other form, $\frac{x}{y}$ can be expressed as '*x* out of *y*', more precisely, $\frac{x}{y}$ can be expressed as '*x* number of parts out of total *y* equal number of parts of something'. In some case, $\frac{x}{y}$ is also used as an operator. For example, when we talk $\frac{3}{4}$ of 100, it means $\frac{3}{4} \times 100 = 75$.

As described above, the term fraction can be used in various forms. Sometimes, the teacher teaches fraction using one particular form and the pupil get confused when they find use of fractions in any other form. Likewise, it can be introduced to the learner in various ways. The basic difficulty before a teacher is how to introduce a fraction—as a rational number or as a ratio or as division of two numbers or as an operator. The main difficulty in teaching and learning the topic is due to the fact that the teachers often remain indulged in choosing the way to introduce the concept and on the other side, the learner gets confused due to its abstract nature and various forms. Among the factors which make rational numbers, in general, and fractions, in particular, difficult to understand are their many representations and interpretations (see Kilpatrick, Swafford and Findell, 2001).

Absence of Natural Frame of Reference

According to Wu (see Wu, 2014) the difficulty in learning fractions is due to absence of a natural reference point and the inherent abstract nature of the concept.

In learning whole numbers, the children always have a natural reference point—their fingers. In case of fractions, it is normally introduced using a pizza or pie or bread and thus, the pizza, pie or bread become the reference points. These may be good reference points to help the beginners for the purpose of the vocabulary learning aspect. However, these models become awkward in case of fractions bigger than 1 or in describing mathematical operations like addition, subtraction, multiplication, etc. For example, one cannot multiply two pieces of a pie (see Wu, 2014; Hart, 2000). If fractions are introduced as numbers (rational numbers), the number line can act as a reference point with help of which the students can visualise the magnitude and position of a fraction, which may ultimately help them in conceptualisation of fractions.

Algebraic Operations on Fractions Create Doubts

Apart from the basic structure of fractions, there are some questions related to the basic arithmetic operations on fractions, which creates doubts in the minds of the learners. For example, while adding or subtracting the fractions, it is necessary to form equivalent fractions in such a way that denominator of all the fractions become equal (equal to the LCM of the denominators), whereas while multiplying or dividing the fractions, there is no such requirement of making equivalent fractions. Further, in case of addition or subtraction, after making the equivalent fractions, only numerators are added or subtracted but denominators are not added or subtracted. Contrary to the practice followed in fraction addition or subtraction, both the numerator and denominator are multiplied in case of fraction multiplication. In case of fraction division, the denominator is reversed and then multiplied with the numerator. Some of the common queries, which come in the mind of the beginners, are as follows.

1. Why $\frac{3}{7}$ and $\frac{2}{7}$ are added as $\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}$, and why it is wrong to say $\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7+7}=\frac{5}{14}$?
2. Why $\frac{3}{7}$ and $\frac{4}{9}$ cannot be added as $\frac{3}{7}+\frac{4}{9}=\frac{3+4}{7+9}=\frac{7}{16}$?
3. Why it is necessary to obtain LCM of the denominators and to make equivalent fractions in case of addition or subtraction of fractions?
4. If $\frac{3}{7}$ and $\frac{2}{7}$ are added as $\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}$, why they are multiplied as $\frac{3}{7}\times\frac{2}{7}=\frac{3\times2}{7\times7}=\frac{6}{49}$ and not as $\frac{3\times2}{7}=\frac{6}{7}$?
5. In case of division of two fractions, what is the logic behind multiplying with the inverse of the fraction in the denominator, i.e., why $\frac{3}{7}\div\frac{2}{7}=\frac{3}{7}\times\frac{7}{2}=\frac{3}{2}$?

6. If a whole number is multiplied with another whole number, the resultant is always greater than the number. However, this practice is not always followed in case of fractions. The result may sometimes be less than the number. Why?
7. Similarly, in case of division by a rational number, the resultant may be greater than the number, unlike in case of the whole numbers where the resultant is always less than the number in case of division. Why?

Though all these queries can be answered mathematically, but these are not normal to understand, do not appear natural and are far from obvious. From the point of view of a beginner, such problems arising during the basic fraction arithmetic are away from realisation, answers to these problems are not apparent and this is one of the important reasons which create difficulty in understanding fractions.

How to Teach Fractions

The previous section described as to why fractions are difficult to understand for the children and what are the challenges before the teachers while teaching fractions to the beginners. The basic structure and some inherent properties of the fractions make it difficult for both teachers as well as children. However, sometimes the method of teaching and the way the fractions are introduced also contribute to make the learning more difficult for the beginners. These difficulties can be overcome and learning can be made easy by changing the techniques of teaching fractions.

Fractions are Basically Numbers and may be Introduced Like that

While teaching whole numbers, the concept of the number line is introduced to the children. The children learn to locate a whole number in the number line, which helps them to conceptualise its position and accordingly its magnitude too. When the position and magnitude of the whole numbers are clearly conceptualised, the children can place the given whole numbers in order. They are able to compare the numbers and place them in ascending or descending order. Some basic algebraic operations are also performed on the number line, which helps the children to understand the logic behind the basic operations in mathematics. Similar practice is required to be followed in case of fractions too. The learners must be able to understand that fractions are also numbers. Further, they must be able to find out the magnitude of the fractions and, accordingly, their position on the number line. Once the positions of the fractions are determined, comparison between the fractions and their order on the number line becomes easy, which may help the children to conceptualise fractions.

The following example may be useful for the teachers to introduce fraction.

Example

When the students become familiar with the numbers, number line, basic operations on number/number line, they may be asked to think what would be there between two whole numbers, say, is there any number which is greater than 5 but less than 6. Similarly, lines having length equal to a whole number can be drawn and children may be asked to find out their length using a ruler. Then, a line of length greater than 4 cm and less than 5 cm may be drawn and the children may be asked to determine its length. This example may provide a thought to the children that there is requirement of numbers other than whole numbers and something may be there between two whole numbers. Several other examples may be formed involving the children. For example, the class may be divided into groups of 4 children each, and each of the groups may be asked to divide 12 toffees or counters equally among themselves. Thereafter, each group may be provided 13 toffees and then they may be asked again to share them equally among themselves. Each group will be left with one toffee which is to be distributed among the four children. With these examples, which can be explained in a practical manner by involving the children, they will start thinking that there is a requirement of numbers other than the whole numbers and there must be numbers between two whole numbers. This may form the background of introducing fractions as rational numbers using the number line.

Procedural vis-à-vis Conceptual Knowledge

Like other topics of mathematics, the knowledge of fractions also has two parts—conceptual knowledge, and working or procedural knowledge. Both are complementary to each other and are essential for proper understanding of the term fraction. Normally, during the teaching of the fractions, more emphasis of the teachers remains on the procedural knowledge so that the children can perform various operations associated with fractions. However, the beginners must be given sufficient time to conceptualise the fractions first. The procedural knowledge may be provided only after the children conceptualise the structure of fraction properly. Researchers have reported that while imparting the knowledge of fractions, procedural knowledge is generally given more importance by the teachers and less or no emphasis is given on the conceptual understanding of the aspect. As a result, the children learn rote procedures and start calculations without understanding the mathematics behind it (Byrnes and Wasik, 1991; Gabriel et al., 2013; Kerslake, 1986). The conceptual and procedural knowledge is complementary to each other and must go simultaneously. The teacher is required to skillfully present a mixture of procedural and conceptual knowledge of fractions to one's students so that they can understand and use it properly.

Comparison of Fractions with the Whole Numbers may be interesting

The teachers must teach fractions like numbers, comparing their properties with the properties of whole numbers to which the students are familiar. It must also be explained to the learners as to why there is a requirement for introducing fractions, what is the significance of fractions, what is the similarity between the fractions and whole numbers and how fractions are different from whole numbers. In addition to this, there are some factors which are also responsible for difficulties faced by the learners. Quality of teaching and the knowledge of the teacher play a pivotal role in minimising the difficulties of the learners. The social, cultural, educational and financial background of the learners as well as of the teacher also makes a great impact on the learning and understanding of the children.

Problems based on Common Sense may be Useful

Once the concept of fractions is properly understood by the children, they can solve some general problems on fractions, without following the prescribed procedure but just applying their common sense. For example, which is greater: $\frac{2}{11}$ or $\frac{17}{19}$? Mathematically, to solve the problem, we need to form equivalent fractions so that the denominator of both the fractions become equal. Then the fractions can be compared. However, in this problem, it can be said with common sense that $\frac{17}{19}$ would certainly be greater than $\frac{2}{11}$. Consider another problem, which is greater: $\frac{6}{7}$ or $\frac{8}{9}$? One method is to make equivalent fraction as is done usually to solve such problems. Alternatively, $\frac{1}{7}$ is required to make $\frac{6}{7}$ complete, i.e., 1 and $\frac{1}{9}$ is required to make $\frac{8}{9}$ complete. Since, $\frac{1}{7}$ is greater than $\frac{1}{9}$, it can be concluded that $\frac{6}{7}$ is less than $\frac{8}{9}$. Such type of common or alternative thinking can help learners to grasp the fractions but it can be developed only when they understand the fractions thoroughly.

Learning should not be Made Abstract

All too often, learners think of 'fractions' as being a discrete (and often difficult) topic that has no real connection with any other area of mathematics (McLeod and Newmarch, 2006). However, meaningful connections make learning more powerful. Work on fractions needs to be integrated into other topics of mathematics; number, shape, data handling, and particularly every sort of measure of weight, length, capacity, time, and simple probability. Learners will encounter fractions throughout their work at all entry levels of the numeracy curriculum, example, solving money problems, sharing a bill, comparing prices, calculating journey times, cooking, interpreting data in pictograms and bar charts, using a meter rule, measuring a room, comparing each other's heights, and checking the weight of ingredients (McLeod and Newmarch, 2006). The teaching of mathematics, particularly when it is related to fractions, decimals and percentage, should not be made abstract. If the teaching is kept abstract and we do not relate it with the day to day activities and surroundings, the children lose interest in study and do not try to understand the phenomenon. Once the rhythm is lost and the children do not understand some of the basic steps, they are not able to grasp the further development. Therefore, the teaching methodology should be designed carefully and be supported by the day to day activities and the environment in which the children are growing.

Various Teaching Tools may be used while Teaching Fractions

Use of teaching tools and audio-visual techniques is important. Developing visual models of rational numbers is critical in building an understanding of multiple and equivalent forms of rational numbers and the relationship among fractions, decimals and percentage (Scaptura and Mahaffey, 2007). It is easy to draw the number line on a paper using a ruler and pencil. It can be divided into equal number of parts to develop the sense of fractions initially. A paper strip may also be used which the children may divide into equal number of parts to understand the number line and the position of various rational numbers or fractions on it.

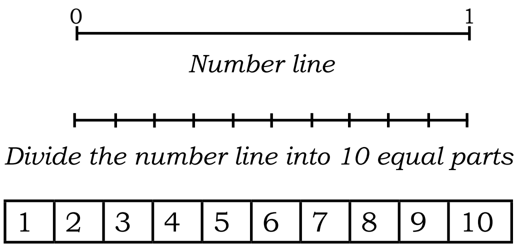

Figure 1. Paper strip as the number line

Conclusion

As suggested in this paper, if the fractions are introduced simply as an extension of the number system, it may be easy for the teachers to teach and the children to follow. If the fractions are introduced using the number line, it will come in a natural way for the beginners as part of the number system. The number line will become the frame of reference, which will help the children to visualise fractions. After understanding the magnitude of fractions and their location on the number line, it becomes easy for the learners to compare two fractions and perform various algebraic operations on it.

If the fractions are taught in the way as suggested in this paper, it will give less load to the beginners towards understanding the structure of fractions and they may be able to focus more towards the operations and properties of fractions. Various difficulties which are being faced by the teachers and the learners while dealing with fractions can be overcome by using the method as suggested in this paper.

References

ARMSTRONG, B.E. AND C.N. LARSON. 1995. Students' Use of Part-whole and Direct Comparison Strategies for Comparing Partitioned Rectangles. *Journal of Research in Mathematics Education.* Vol. 26, No. 1. pp. 2–19.

BEHR, M.J., R. LESH, T.R. POST, AND E.A. SILVER. 1983. Rational Number Concepts. In M. Landau (Ed.), *Acquisition of Mathematics Concepts and Processes.* pp. 91–126. Lawrence Erlbaum Associates. Hillside, NJ.

BYRNES, J.P., AND B.A. WASIK. 1991. Role of Conceptual Knowledge in Mathematical Procedural Learning. *Dev. Psychol.* Vol. 27. pp. 777–786. doi: 10.1037/0012-1649.27.5.777

CLARKE, D. AND A. ROCHE. 2010. The Power of a Single Game to Address a Range of Important Ideas in Fraction Learning. *Australian Primary Mathematics Classroom.* Vol. 15, No. 3. pp. 18–24.

CLARKE, D., A. ROCHE AND A. MITCHELL. 2011. Year Six Fraction Understanding: A Part of Whole Story. In J. Watson and K. Beswick (Eds), *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australia.* Adelaide, Australia: MERGA Inc. Vol. I. pp. 207–216.

CONDON, C. AND S. HILTON. 1999. Decimal Dilemmas. *Australian Primary Mathematics Classroom.* Vol. 4, No. 3. pp. 26–31.

EMPSON, S. AND L. LEVI. 2011. Extending Children's Mathematics: Fractions and Decimals: Innovations in Cognitively Guided Instruction. NH: Heinemann, Portsmouth. pp. 178–216.

ERLWANGER, S.H. 1973. Benny's Conception of Rules and Answers in IPI Mathematics. *Journal of Children's Mathematical Behaviour.* Vol. 1, No. 2. pp. 7–26. Reprinted in Carpenter, T.P., Dossey, J.A., and Koehler, J.L. (Eds) 2004. Classics in Mathematics Education Research. pp. 48–58. National Council of Teachers of Mathematics. Reston, VA.

FAZIO, L. AND R. SIEGLER. 2011. *Teaching Fractions – Education Series 22.* UNESCO & International Bureau of Education, Gonnet Imprimeur, 01300 Belley, France.

FUCHS, L.S., R.F. SCHUMACHER, J. LONG, J. NAMKUNG, C.L. HAMLETT, P.T. CIRINO AND P. CHANGAS. 2013. Improving At-risk Learners' Understanding of Fractions. *Journal of Educational Psychology.* Vol. 105, pp. 683–700. doi: 10.1037/a0032446

FUCHS, L.S., R.F. SCHUMACHER, S.K. STERBA, J. LONG, J. NAMKUNG, MALONE, AND A., P. CHANGAS. 2014. Does Working Memory Moderate the Effects of Fraction Intervention? An aptitude–treatment interaction. J*ournal of Educational Psychology.* Vol. 106. pp. 499–514. doi: 10.1037/a0034341

Gabriel, F., F. Coche, D. Szucs, V. Carette, B. Rey and A. Content. 2013. A Componential View of Children's Difficulties in Learning Fractions. *Frontiers in Psychology.* Vol. 4, pp. 1–11. doi: 10.3389/fpsyg.2013.00715

Hart, K. 2000. Mathematics Content and Learning Issues in the Middle Grades. In Mathematics Education in the Middle Grades: Teaching to Meet the Needs of Middle Grades Learners and to Maintain High Expectations. *Proceedings of a National Convocation and Action Conferences.* pp. 50–57. National Academy Press, Washington, USA.

Hecht, S.A. and K.J. Vagi. 2012. Patterns of Strengths and Weaknesses in Children's Knowledge about Fractions. *Journal of Experimental Child Psychology.* Vol. 111, pp. 212–229. doi: 10.1016/j.jecp.2011.08.012

Jordan, N.C., Hansen, N., Fuchs, L.S., Siegler, R.S., Gersten, R. and D. Micklos. 2013. Developmental Predictors of Fraction Concepts and Procedures. *Journal of Experimental Child Psychology.* Vol. 116, pp. 45–58. doi: 10.1016/j.jecp.2013.02.001

Kamii, C. and F.B. Clark. 1995. Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behaviour.* Vol. 14, No. 4. pp. 365–378.

Kerslake, D. 1986. Fractions: Children's Strategies and Errors. *A Report of the Strategies and Errorsin Secondary Mathematics Project.* NFER-NELSON Publishing Company Ltd. England.

Kieren, T.E. 1980. The Rational Number Construct: Its Elements and Mechanisms. In T.E. Kieren (Ed.). *Recent Research on Number Learning.* ERIC Clearinghouse for Science, Mathematics and Environmental Education. pp. 125–149. OH, Columbus.

Kilpatrick, J., J. Swafford and B. Findell. (Eds). 2001. Adding it Up: Helping Children Learn Mathematics. National Academy Press. Erlbaum. Washington, DC. Mahwah, NJ.

Lortie-Forgues, H., J. Tian and R.S. Siegler. 2015. Why is Learning Fraction and Decimal Arithmetic so Difficult? *Developmental Review.* Vol. 38. pp. 201–221. doi:10.1016/j.dr.2015.07.008.

Ma, L. 1999. Knowing and Teaching Elementary Mathematics: Teachers' Knowledge of Fundamental Mathematics in China and the United States. *Studies in Mathematical Thinking and Learning Series.* Routledge, NY. 10017.

Mack, N.K. 1990. Learning Fractions with Understanding: Building on Informal Knowledge. *Journal for Research in Mathematics Education.* Vol. 21, No. 1. pp. 16–32.

McLeod, R. and B. Newmarch. 2006. *Fractions, Math for Life.* National Research and Development Centre for Adult Literacy and Numeracy. London.

Moss, J. and R. Case. 1999. Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education.* Vol. 30, No. 2. pp. 122–147.

National Council of Teachers of Mathematics. 2007. *The Learning of Mathematics: 69th NCTM Yearbook.* National Council of Teachers of Mathematics, Reston, VA.

Orpwood, G., L. Schollen, G. Leek, P. Marinelli-Henriques and H. Assiri. 2012. College Mathematics Project 2011: Final report. Retrieval from http://collegemathproject.senecac.on.ca/cmp/en/pdf/FinalReport/2011/CMP_2011_Final_Report%20-%2002Apr12%20pmh.pdf

Perle, M., R. Moran and A.D. Lutkus. 2005. *NAEP 2004, Trends in Academic Progress: Three Decades of Student Performance in Reading and Mathematics.* National Center for Education Statistics. Washington, DC.

Post, T. R., K.A. Cramer, R. Lesh, G. Harel, and M. Behr. 1993. Curriculum Implications of Research on the Learning, Teaching and Assessing of Rational Number Concepts in T.P. Carpenter, E. Fennema and T.A. Romberg (Eds), Rational numbers: An Integration of Research. pp. 327–362. Lawrence Erlbaum Associates, Hillsdale NJ.

Scaptura, C., J. Suh and G. Mahaffey. 2007. Masterpieces to Mathematics: Using Art to Teach Fraction, Decimal and Percent Equivalents. *Mathematics Teaching in Middle School.* Vol. 13, No. 1. pp. 24–28.

Siegler, R.S. and A.A. Pyke. 2013. Developmental and Individual Differences in Understanding of Fractions. *Developmental Psychology.* Vol. 49. pp. 1994–2004. doi: 10.1037/a0031200.

Stigler, J. W., K.B. Givvin and B. Thompson. 2010. What Community College Developmental Mathematics Students Understand about Mathematics. *The Math AMATYC Educator.* Vol. 10, No. 4–16.

Wu, H. 2014. *Fractions, Decimals and Rational Numbers.* Report of the Learning Processes Task Group of the National Mathematics Advisory Panel, Berkeley. doi: https://math.berkeley.edu/~wu/NMPfractions.pdf