Facets of Mathematics Education towards Achieving Mathematics Competence in Children

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Abstract

Educational reformations around the globe have been structured to address challenges in the attainment of learning objectives that surpass content and topic characterisation. The notion of mathematical competency signifies such objectives. In spite of having variations in the idea and definition of mathematical competency, one common construct is to have an ability to apply mathematics to solve problems in everyday situation. Many international mathematics assessments have provided a platform to children from different countries to get them assessed and know how much competent they are in solving unseen mathematical problems in everyday situations. There is a great need to identify which mathematics competencies a child should have and what the issues and challenges in achieving these competencies are. The idea of these competencies cannot be perceived in isolation. It should be backed up by strong theoretical framework and other integral aspects of mathematics education that are vital as per various studies. The aim of this paper is to explore the idea of mathematics education based upon building mathematical competencies among children and an assessment commensurate to that which may help policymakers in achieving the goal of preparing mathematically competent children who are able to apply mathematics to solve real life problems.

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Introduction

Mathematics teaching and learning of in the last few years has been evolved to be targeted towards achieving mathematics competencies by national policymakers and individual researchers or research establishments. The works of many researchers (for example, Niss and Jensen, 2002) have contributed significantly towards the enrichment of knowledge in this domain. Organisations across the world (for example, OECD, 2009) have been engaged in formulating various mathematical competencies and their assessment. Methods of knowledge description and educational goals are also listed in Trends in International Mathematics and Science Study (Mullis, Martin, and Foy, 2005) and Programme for International Student Assessment (OECD, 2009) that have resulted in major policy revisions in many countries (Pettersson, 2008).

Justification problem is one of the important problems in mathematics education that is suffering from 'relevance paradox'. While many think mathematical knowledge as relevant for the society as a whole, another group of people questions about its relevance to them as an individual being. Traditionally, international patterns are characterised as 'mathematics for all' but the trend has been questioned by different agencies. Societies have not been fully equipped with mathematics knowledge by education providers and necessary competencies were not built among them (Niss, 2003).

Mathematics is seen as an obstruction on by some due to fear and despair to find solution to problems (Benson-O'Connor, McDaniel, and Carr, 2019). It is difficult to change well set principles in the classroom and most teachers teach the way they have been taught (Mosvold, 2005).

The problems related to the quality of mathematics teaching have also been tried to be addressed by agencies in India. NCERT (NFGTM, 2006), through its position paper on mathematics, has highlighted certain key points in mathematics teaching and learning. Mathematising of children is a key point suggested in it and many other important areas of attention have also been suggested in the document. It is pertinent to know to what extent these suggestions are in conformity with mathematics competencies of the children of Indian schools.

Mathematics competencies have been broadly discussed in various policy papers and research documents and most of them associate these competencies with the children's ability to use their mathematical knowledge in solving real life contextual problems in everyday situation. So the competencies have to be weighed according to this perspective also while framing policies.

Mathematisation

The origin of the term 'mathematisation' is not known but the concept of mathematisation has been adopted and supported by many researchers. While Freudenthal

(1991) considered mathematising as 'the process by which reality is trimmed to the mathematician's needs and preferences and that continues as long as reality is changing, broadening and deepening under a variety of influences, including that of mathematics, which in turn is absorbed by that changing reality.' According to him, mathematising as a term' might have been resulted from informal talks and possibly preceded by the terms like 'axiomatising, formalising and schematising'. His interpretation of the term 'mathematisation' was a product of his description of mathematics as an activity of raising the level of certainty where modes of raising the lower level become significant for leaving at upper levels. He advocated for the importance of capability of a child to identify the problem in a situation.

'Mathematisation' as an idea was further enriched by Treffers, A. 1987 Three Dimensions, a Model of Goal and Theory Description in Mathemacties Instraction The Wiskobas project. Dordrecht, the Nether and Reidel Publishing Companey who proposed a distinction between horizontal mathematisation and vertical mathematisation through his design practices. The aim of vertical mathematisation is to raise the understanding of children by developing a mathematical model through problem symbolisation and organisation that allows children to provide orientation through beginning from a paradigmatic situation. Horizontal mathematics focuses on context organisation and preparation for mathematical treatment.

National Council of Educational Research and Training, in its National Curriculum Framework 2005 (NCERT, 2005), suggested that students' focus while learning mathematics should be on the aspect of mathematisation and developing this should be the primary aim of mathematics teaching in schools. It suggested new mathematics teaching practices considering mathematisation as a broad aim of education. The suggestion was commensurate to the idea of Wheeler who opined that knowing the way of mathematisation development among learners is far more required than knowing mathematics itself (NFGTM, 2006).

So, mathematisation should be at the core of mathematics learning processes. It can be promoted by the classroom interactions of teachers and students. Many mathematics education models stressed on mathematisation rather than rote memorisation of concepts. Realistic Mathematics Education (RME) model has its roots in Hans Freudenthal's interpretation of mathematics as a human activity (Freudenthal, 1991). To this end, Freudenthal accentuated that it is more important to reinvent mathematisation rather than doing mathematics. Gravemeijer (1994) states that realistic education in mathematics is supported by 'guided reinvention, progressive mathematisation through phenomenological exploration and self-developed models.' The learner, the teacher and the curriculum content — all play their part in the process which is characterised by mathematisation of learners' thoughts through careful guidance by the teacher, of all the activities done by the learner. The student is not told exactly what to be done but he/she is guided so that mathematics may be reinvented by themself by exploration and establishment of connection between the ideas. Concepts are developed as abstract ideas from informal thoughts of the learners through this process involving mathematisation.

Mathematical Literacy

According to OECD (1999), real mathematics learning by an individual gets reflected in their competency to understand and recognise the use of mathematics in the world, to make prediction based on mathematical foundations and to engage in mathematics in a manner that fulfils the requirements of their present and future life as a careful and thoughtful citizen, who can construct and reflect upon one's thoughts. 'Mathematical literacy' here not only includes the traditional knowledge and skills but also applying these in multiple contexts in multiple ways. The context should have an element of reflection and insight. Thus, mathematical literacy cannot be restricted to acquiring knowledge but it requires how the factual knowledge and skill has to be used to implement the procedures. 'The world' represents the society and the culture where one lives. As Freudenthal 1991 states that all natural, societal, psychological phenomena in this world are controlled by mathematical theories, edifices and notions. The word 'engaged' not only means activities of physical or social involvement, but it also includes many other aspects like ascertaining, standing behind, being vocal towards various elements of mathematics. 'Present and future life' stands for personal and professional life with friends and relatives as well as life as a community member. Thus mathematical literacy pertain not only having the knowledge of formulae and procedures but also the capacity of using mathematics in many types of situations. These situations may be solving a problem using mathematical knowledge, formulae, results and theory, and also those situations which require how to convert the problem into a mathematical problem. The identification of mathematical structure involved in the solution has to be done by the one who wants to pose or solve the problem. Capability of solving mathematical problems together with the ability to formulate and pose problems in various domains and situations is an integral requirement of mathematical literacy.

Characterisation of Mathematical Literacy

Mathematics literacy as per OECD (1999) is characterised by primary and secondary attributes. The primary attributes comprise mathematical competency and mathematical themes, and the secondary attributes comprise mathematical curricular strands, and situations and contexts. The primary attributes characterise how proficient the learner is and the secondary aspects give a description of the balance between tasks and ensure that the domain is adequately covered. 'Mathematical themes' and 'mathematical curricular strands' together describe the content of mathematics.

'Mathematical competencies' are those that give an insight into how much proficient a learner is in solving mathematical problems, how much competent a learner is in using mathematical language and how much perfect a person is in mathematical modelling.

Themes in Mathematics are so called because they do not mean division of mathematics into distinct topics but they are groups of related mathematical notions that emerge in various contexts and real life situations, for example, chance, geometry, change, relationship, etc.

'Mathematical curricular strand' signifies classroom mathematics content taught in schools, for example, geometry, sets, statistics, etc.

The other secondary attribute is that of contexts and situations in which mathematical problems are posed, for example, professional, personal, social and academic or educational context.

Mathematical Competencies

This is the first major characteristic of mathematical literacy structure. This structure is a set of mathematical skills admissible and appropriate to all strata of mathematics students. There is no hierarchy in these skills. The under mentioned eight skills belong to the set.

Mathematical Thinking

It means characterisation of mathematical problems through the ability to pose a question and to have the knowledge of answers offered by mathematics; familiarity with various types of statements like 'definitions', 'theorems', 'conjectures', 'hypotheses', 'examples' and 'conditioned assertions', etc. and also having the capacity of assessing limit and extent of a mathematical concept.

Mathematical Argument

It is the ability of knowing and understanding the proofs of mathematics results and differentiating them from other types of reasoning; using and evaluating the sequence of various kinds of logics and reasoning; building a sense of heuristics and analysing the statements logically.

Modelling

It consists of structuration of the problem to be modelled, infusing 'reality' into mathematics and its structure through mathematisation, describing mathematical models using the mirror of reality via de-mathematisation, trying various mathematical models and doing their validation, critically reviewing mathematical model and outcomes achieved, drawing conclusions on the basis of modelling and results, and supervision and governance of the modelling process.

Problem Posing and Solving Skill

It consists of asking, framing and describing various types of mathematical questions as well as providing their multiple solutions.

Representation

It consists of unscrambling, inferring and differentiating, among various types of representations, of mathematics based conditions and entities and also interdependence among them. It also includes the skill to choose from and switch over various forms of representation as per the objective and the situation.

Symbolic, Formal and Technical Skill

It consists of unscrambling and inferring symbols based abstractions and ability to understand its relation with common language, expressing linguistic mathematical thoughts through signs and symbols, working with mathematical statements that contain formula or symbols, identification of variables, solving equations and performing calculations.

Further enrichment of this skill among learners requires them to express themselves or understand what others wish to express in a variety of innovative and creative ways like written, verbal, visual, etc. By adopting such means, the abstract and symbolic content may be better delivered or understood by the learners. For example, venn diagrams in set theory, pie charts, graphs in problems based on distance, speed or acceleration, timeline for problems like growth of microorganisms/human population and number line to understand the behaviour of real numbers, etc. The symbolic, formal and technical skill is linked with this additional skill named as 'communication skill' in the sense that the former is about knowing, understanding and expressing mathematical thoughts through formal mathematical symbols and abstraction, while the latter deals with understanding mathematical thoughts that others tried to communicate by adopting multiple means and also organising and presenting the abstract symbolic ideas in a form that may be well understood by others.

Communication Skill

It consists of the ability to express mathematical thoughts in mathematical language in multitude of directions and ways, and also the ability to understand such mathematical languages in which others expressed their mathematical content and viewpoints and the languages may be both verbal and written.

Aids and Tools

It consists of familiarity with various instruments and technological and physical aids, software, etc., that provide assistance in mathematical activity and also with the constraints of such aids and instruments. It is considered as a skill because it involves knowing which tool, software or aid should be used for a specific problem, using it what are the limitations of their use, what are the constraints, etc., and also reflective use of these tools, software or aids in solving a mathematics problem. For example, a spreadsheet may be helpful when the aim is to see the effect of multiplication of decimal numbers by 0.01. A graphical calculator may help in understanding that quadratic equations like $x^2+x+1=0$ have no real root. Equations such as $x^2–2x–1=0$, whose roots are irrational numbers, can be observed through graphic tools by noting down the points of intersection of the parabolic graph with x-axis. Cabrilog is useful for school geometry. Excel and SPSS are useful for statistics. Mathigon and Wolfram Alpha are good resources to understand a few topics in school mathematics.

Competency Levels

According to OECD (1999), assessment of the above mentioned skills should not be made individually. While applying mathematics to real life situations, usually many and perhaps all of the skills are used simultaneously. So, the process of developing test items that aim to assess individual mathematical skills may lead to a false partitioning of the area of mathematical literacy.

For the purpose of assessment of the aspects of mathematical competencies through development of tests and items, skills are proposed to be organised into three broad competency levels which may be characterised as follows:

- **Level 1:** Defining various mathematical terms, ability to reproduce mathematical facts and calculation skill come under this level.
- **Level 2:** Capability of connecting real life problems with mathematics and integrating mathematical information for solving them is part of this level.
- **Level 3:** Ability of generating mathematics based thoughts, generalising basic mathematical facts and having an insight of solving a problem using mathematics is included under this level.

All the skills discussed earlier are expected to affect all these three competency levels. Those skills cannot be categorised into a single competency level. There is a 'conceptual continuum' formed of these levels spread over from the ability of knowledge reproduction and computations to the ability of connecting different components of mathematical activities aimed at solving real life mathematics problems.

Thus the levels are hierarchical in nature in a manner that a series of goals to be achieved in Level 3 competencies is tougher than the series of goals to be achieved in Level 2 competencies. But this does not mean that Level 2 competencies are necessary for achieving Level 3 competencies. A few studies (De Lange, 1987; Shafer and Romberg, 1999) emphasised that it is not compulsory to attain perfection in Level 1 competencies to achieve Level 2 or Level 3 competencies.

To understand it more clearly, consider the following example.

Example: You want to buy 150 juice bottles for a party. How many eight-bottle packs do you need to buy?

Even if a learner is not perfect in Level 1 competency and is not able to perform division, still by using one's own approach he/she may discover the answer. This means he/she could connect the problem with mathematics without knowing the area of mathematics to which the problem is related. Thus he had a good Level 2 mathematics competency of connection and integration without having enough Level 1 competency. One such approach could be given ratio table method below:

1	10	5	15	2	17	2	19
8	80	40	120	16	136	16	152

Thus tasks become more difficult as we progress from Level 1 to Level 3, but a person may achieve a higher level competency even without having lower level of mathematics competency. But sometimes in higher level competency, tasks related to a lower level competency are also involved and so it is important to achieve competency in all the three levels.

Example: An auto charges a fixed amount of ₹25 for first 2 kms and thereafter it charges a fixed amount per km (less than 500 m are ignored by the bill meter and 500 m or more is rounded to an extra km). You paid a bill of ₹265 for a distance of 25.650 km. What is the fixed charge per km?

In the above example, having Level 2 competency may help learners to formulate the problem into a one variable linear equation (such as $25+24x = 265$), but equally important is Level 1 mathematics competency that helps them to solve it and arrive at the answer. So connection and reproduction both are required here.

Mathematical literacy as proposed by OECD (1999) focuses on students' capacity building in undertaking activities that require skills in the three competency levels. Thus, assessment assignment should be composed of by assigning weightage to all three competency levels so as to provide opportunity to policymakers to judge wellness of their schools and curricula as regards the three competency levels.

Level 1 Competencies: Reproduction, Definitions and Computations

This level consists of factual understanding, symbolisation and expression, ability to differentiate and establish equivalences, familiarity with mathematical facts entities, performing mathematical procedure, having idea about how to apply basic algorithm and proper technical skills, ability to manipulate symbolic mathematical language and computations. Test items meant for the assessment of Level 1 competency should be MCQ type or partly subjective type.

Examples

1. Solve the equation $2y + 5 = 3 - y$.
2. The longest chord of a circle is called ________.
3. What is the area of a rectangle with sides measuring 5 cm and 3 cm?
4. What is the factorisation of $x^3 - 8$?
5. Locate the points (–3, 6) and (3, 6) on a graph paper.

Level 2 Competencies: Connections and Integration for Problem Solving

The competencies in this level aim at connecting various fields and areas of mathematics and integrating the knowledge for solving elementary level mathematical problems. The learners are encouraged to exercise the choice of strategy and suitable mathematical tool to solve the problems. A lower degree of mathematisation is required to solve problems in this competency level that are classified as 'non-routine'. Students are hoped to be able to use various techniques of representation as per the motive and requirement of the situation. The aspect of connecting mathematical problems expects the learners to have an ability of differentiating and establishing relationship between statements like 'definitions, claims, examples, conditioned assertions and proofs'.

This level is related to the above mentioned skills. Solutions to problems in the example below require acquaintance with 'mathematical argumentation skills' because a little bit of reasoning and argumentation is needed to obtain the answers to the problems listed. Modelling skill is also required in this competency level and this can be observed in the example. The task of problem solving itself requires 'problem posing and problem solving skill'. 'Representation skill' is used by the students when they use different modes of mathematical representation like chart, graph, tables and diagram, etc.

Language is an important aspect of mathematical literacy components. Describing and decoding symbolic and formal language, and studying their relationship with natural languages are also imperative skills included in this competency level. Items included in the assessment of competencies in this level are usually based on a context and students are engaged to understand the context and take decisions accordingly. From the following example of this level, one may observe that, unlike Level 1 example, neither the mathematical curriculum domain to which the example belongs nor the algorithm, method or technique that is the best way to solve the problem, In fact, at times, curricular strand depends upon the strategy choice by students and there may be multiple strategies suitable for a problem.

Examples

1. You started the month of June with a full bottle of edible oil in your kitchen. After 20 days, the bottle remains only a quarter full. Will there be a shortage of the oil before expiry of the month? (Assume that the same pattern of consumption continues during the last ten days).
2. You went to buy a shirt in a shop. There were two schemes of discount—. Scheme 1: Flat 30% off, *Scheme 2:* 20%+10% off.

Which scheme will result in more saving?

Level 3 Competencies: Mathematical Thinking, Generalisation and Insight

This competency level requires learners to be assessed on the basis of test items in which students are expected to 'mathematise' situations. They are expected to identify and source mathematics rooted in these situations, to apply mathematics to provide solutions to problems, to be engaged in analysis and interpretation, to build their own models, to prepare strategies, and to develop mathematical arguments through proofs, deductions, generalisations, etc. Model analysing and reflecting on the process are also the competencies included in this level. Posing problems along with solving them is significant to achieving competencies in this level. An adequate and effective communication is a key for achieving competencies in this level. Communication may be in multiple forms like written, verbal and visual, etc., and is a two-sided process. This means that the learners should be able to communicate their own thoughts and ideas and also they should be competent to understand what others are willing to communicate. Lastly, the learners should be able to enquire about many other important aspects of mathematics. They should be able to have an understanding about the inherent characteristics of mathematics that include cultural aspects of the subject as well as history associated with its different concepts. Applying mathematics to unscramble contextual problems and problems arising in other discipline areas is also significant for achieving competencies in this level. Considerable perfection in mathematical modelling of such problems is expected to promote these competencies.

Tasks to be executed for the assessment of competencies in this level often include competencies and skills from other two levels also. This level, though is the most hard to be assessed, yet is a key component of mathematical literacy. The construction of items in the assessment of competencies in this level relies, in terms of suitability, on extended response questions with multiple answers instead of multiple choice questions. But designing of such items along with the evaluation of students' responses is not an easy process. Still, since this level is an important level for achieving mathematical literacy, so effort should be made to include items from this level, even though in a limited number.

Example

A pizzeria shop sells small, medium and large size pizzas with diameter 4, 6 and 8 inches, respectively, the mentioned in its menu card along with their prices— ₹100, ₹200 and ₹300, respectively. You want to spend a total of ₹500 to buy pizzas. How many pizzas of each size will you buy to get a better deal? Assume that all pizzas sold by the pizzeria have the same thickness.

One aspect that rarely gets noticed in the mathematisation process is the necessary step of pondering upon and analysing the solution of mathematical contextual problems. It can be said that every competency level involves mathematisation because in every contextual problem various mathematics concepts has to be applied. But mathematisation involved in Level 3 competencies is particularly important and it goes beyond merely identification of important mathematics involved in a problem. Below given are two examples. The first one involves Level 2 competency and simple mathematisation. Second example is based on Level 3 competency as it involves a little complex mathematisation. It expects the student to identify the relevant mathematics and also to formulate and communicate a mathematical argument.

Examples

1. You need to cut a thread of length 16 cm into two pieces such that the length of one piece is thrice the length of the other piece. What are the lengths of the two pieces?
2. Suppose the government of a country bought wheat from farmers in the year 2020 at the rate of ₹1800 per quintal and in the year 2021 at the rate of ₹2000 per quintal. Inflation from the year 2020 to the year 2021 amounted to 12 per cent. Government sold the wheat to general public at buying price.

   (a) You are the minister of food in your country and you need to attend farmers' conference. How will you convince them that buying price has been increased by the government?

(b) After few days you have to address a public rally. How will you convince the public that wheat selling price has been decreased?

Pedagogical Aspects of Achieving the Competencies

Mathematical literacy is important for every student to do well in most of the fields—academic or professional. Effective mathematisation of students' thoughts by the teachers is necessary to achieve this literacy during curriculum and instructions delivery. Mathematisation enables the learners to think and visualise the situations through the lens of mathematics. The important components of mathematisation are identification of the contextual problem, organising it using concepts in mathematics, conversion of the real problem into a mathematical one, solving the mathematical problem so obtained using mathematics and context specific interpretation of the solution obtained, all of which help the learners to solve the problem by mathematising the situation. It may not be an easy task for students to formulate the given real life problem mathematically and interpret the mathematical solution into useful information. Consistent efforts should be made by the teachers to simplify the process of mathematisation among students. The following strategies are important towards an effective classroom transaction directed towards Mathematical Literacy through mathematisation of children's minds.

Mathematics can be seen an a language a language too and understanding this behaviour of mathematics is important for mathematisation. A student may easily express a given situation mathematically if he/she can recognise patterns by observing connection, structures, similarity, rules, etc. The role of a teacher is helping students to look for patterns in various daily life situation Language of mathematics proves crucial at the next stage in which the identified pattern is represented using the mathematics symbols, notations and rules, etc. A teacher may help the students understand these notations, rules and symbols, etc., through different examples.

A connection based mathematics learning environment with an element of independent thinking and exploration may be created by teachers. Students should be engaged in real life situations where they learn to discover and extract mathematics and a teacher should help them in filling the learning or knowledge gap, if any, . This integrated approach may help children in achieving necessary mathematics skills and competencies.

Conclusion

Mathematical competencies, if built among students, have the potential to prepare a level of students who are competent enough to apply mathematics to real life problems and are truly compelent in mathematics. These competencies do not work in isolation but many of them are required together to perform a mathematical task. Depending upon the difficulty of mathematisation level involved and of the mathematical task to be performed, different mathematical competencies may be categorised into various non-hierarchical levels such that mathematical task to be done for assessing the Level 3 competency is more difficult than that for assessing Level 2 competencies. Also items meant for assessing Level 3 competencies involve more complex mathematisation than the level of mathematisation involved in items meant for assessing Level 2 competencies.

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