Engineering in Mathematics Education

Mathematical Engineering

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Abstract

We all have our own experiences of Mathematics since our very early childhood. Most of us have developed our own understanding of learning of mathematics. Through this article, the author appeals for a self-analysis of our understanding of learning and teaching Mathematics. The focus of this article is largely devoted to supporting the improvement of mathematics teaching and learning and ultimately the performance of students on measures of mathematics achievement. This article is written with the hope that it will help the reader understand how research-based strategies can support the engineering of positive change to the structures supporting the teaching and learning of mathematics in educational settings. Basically, main emphasis will be on the engineering set up in mathematics education. The entire findings are based on discussion on mathematics of concept of "fraction".

"Mathematics—I want to say—teaches you, not just the answer to a question, but a whole language-game with questions and answers".

—Ludwig Wittgenstein

If someone were to write about "How chemists can contribute to chemical engineering", that person would be considered a crank for wasting ink on a non-issue. Chemical engineering is a well defined discipline and chemical engineers are perfectly capable of doing what they are entrusted to do. They know that chemistry need for their work. Therefore, what we are going to discuss of "How mathematicians can contribute to Higher Secondary School Mathematics education in terms of above objectives." In other words, we will discuss how mathematicians will perform their role in engineering set up in mathematics education and will elaborate "Mathematical Engineering". We will discuss this matter with the presentation of "fraction" concept. It is an attempt to put in perspective the detailed description of the basic skills and concepts in learning and teaching of mathematics through the illustration of fractions. However, an entirely analogous discussion of customisation can be given

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to any aspect of mathematics education.

"*Knowledge and productivity are like compound interest. Given two people of approximately the same ability and one person who work 10% more than the other, the former will more than twice out produce the later. The more you know, the more you learn; the more you learn, the more you can do; the more you can do, the more the opportunity — it is very much like compound interest. I don't want to give you a rate, but it is a very high rate. Given two people with exactly the same ability, the one person who manages day in and day out to get in one more hour of thinking will be tremendously more productive over a lifetime.*" — Richard Hamming

How can mathematics educators be more productive teachers? How do we accelerate students' learning of school mathematics? These are difficult questions. The teaching and learning process is embedded in a complex web of schools, communities, and state governance systems that each play a role in expanding students' opportunity to learn and think about mathematics.

The National Council of Educational Research and Training (NCERT) have developed *National Curriculum Framework* (NCF) — 2005. In NCF-2005, the two goals 'narrow aim' and 'higher aim' of mathematics education have been characterised. By higher aim, we mean to develop the child's inner resource to think and reason mathematically, logical conclusion and handle abstraction. While by narrow aim, we mean that child would have very good algorithmic practices by just remembering the formulas. We are ambitious in the sense that our learning mathematics should achieve this higher aim rather than only the narrow aim.

This distinction of 'narrow aim' and 'higher aim' was first made by George Polyà — a great mathematician as well as a great mathematics educator; he wrote, more generally, that the aim of education should be to develop the inner resources of the child. Here are some quotes of Polyà's:

> "Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures."

But I think there is one point which is even more important. Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims, about which I am now talking are some general tactics of problems — to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school but more complicated problems of engineering, physics and so on, which will be further developed in the higher classes. But the foundations should be started in the primary school. So, I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

> "There are as many good ways of teaching as there are good teachers. But let me tell you what my idea of teaching is. Perhaps the first point, which is widely accepted, is that teaching must be active, or rather active learning. That is the better expression."
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> "You cannot learn just by reading. You cannot learn just by listening to lectures. You cannot learn just by looking at movies. You must add from the action of your own mind in order to learn something. You can call this the Socratic method, since Socrates expressed it two thousand years ago very colorfully. He said that the idea should be born in the student's mind and the teacher should just act as a midwife. The idea should be born in the student's mind naturally and the midwife shouldn't interfere too much, too early. But if the labor of birth is too long, the midwife must intervene. This is a very old principle and there is a modern name for it — discovery method. The student learns by his own action. The most important action of learning is to discover it by yourself. This will be the most important part in teaching such that what you discover by yourself will last longer and be better understood."
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> "This is the general aim of mathematics teaching — to develop in each student as much as possible the good mental habits of tackling any kind of problem."
>
> "You should develop the whole personality of the student and mathematics teaching should especially develop thinking. Mathematics teaching could also develop clarity and staying power. It could also develop character to some extent but most important is the development of thinking."

My point of view is that the most important part of thinking that is developed in mathematics is the right attitude in tackling problems and in treating problems. We faced problems in everyday life like science, politics etc. The right attitude to thinking is maybe slightly different from one domain to another, but we have only one head, and therefore, it is natural that in the end there should be just one method to tackling all kinds of problems. My personal opinion is that the main point in mathematics teaching is to develop the tactics of problem solving.

In these few quotes, Polyà has said something of great significance to mathematics educators.

In fact, the twin concerns of the mathematics education are to engage the mind of every student and also to strengthen the student's resources. I definitely believe that while teaching and learning, whenever you got the actual sense of some tedious concept you might have felt a pleasure and confidence. In our mind, there is a model of learning that informs us and affect whatever we plan in our lesson. Let us see following example of communication in class:

- Some teacher may believe that children enjoy repeating over and over again and being told the correct procedure to be followed.
- Another person may believe that children must know the answer to all the different exercises and must also know the best and shortest method to solve certain problems.
- Yet another mathematics teacher may believe that it is important to allow children the opportunity to solve problems and talk about how they have solved them.

Thus, there are several ways of looking at learning.

NCF–2005 has recommended shifting the focus of Mathematics learning from achieving 'narrow' goals to 'higher' goals. The content areas of Mathematics addressed in our schools do offer a solid foundation. The major challenge which is in front of us is related to Mathematical Process. That is, we have to emphasis the beauty of Mathematics as problem solving, Mathematics as communication, Mathematics as reasoning, Mathematical connections, Use of Patterns, Visualisation, Estimation and approximation.

Giving importance to these processes constitutes the difference between mathematisation of thinking and memorising formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.

To maintain the above mentioned original essence and power of learning mathematics. Mathematicians like H.Wu. (2006), Bass (2005) have made a vision on Mathematics Education as Mathematical Engineering.

The engineer as a metaphor representing a change agent requires a brief explanation. To some, the engineer may appear to be synonymous with the scientist. The distinction between a scientist and engineer is partially clarified by examining two activities related to the preparation of each professional-analysis and design. In science classes, students are required to answer problems, observe phenomena in laboratory settings, record observations, and perform calculations. This process is the essence of analysis. In engineering classes, the instruction often stresses the importance of design. The difference between analysis and design can be described in the following way: If only one solution to a problem exists, and discovering it merely entails putting together pieces of discrete information, the activity is probably analysis (Horenstein, 2002). In comparison, if more than one solution exists and if determining a reasonable path demands being creative, making choices, performing tests, iterating, and evaluating, then the activity is design. Design often includes analysis however; it also must involve at least one of these latter components.

Mathematics education is mathematical engineering. It is not an analogy. It is not used 'engineering' as a metaphor. Rather, a precise description of what mathematics education really is as follows:

One meaning of the word 'engineering' is the art or science of customising scientific theory to meet human needs. Thus, chemical engineering is the science of customising chemistry to solve human problems, or electrical engineering is the science of customising electromagnetic theory to design all the nice gadgets that we have come to consider indispensable. For example, Chemical engineering put chemistry for the plexi-glass tanks in aquariums, the gas we use in our car, shampoo, Lysol, etc. Electrical engineering put electromagnetism in computers, power point, iPod, lighting, motors, etc.

Striking Example of Electrical Engineering

In 1956, IBM launched the first computer with a hard disc drive. The hard drive weighed over a ton and stored 5MB of data. Today's hard drives weigh only a few ounces and hold 100,000 times as much data. These hard drives are built on the same scientific principles. But 50 years of continuous engineering have created refinements that make them enormously better adapted to the needs of consumers.

It will put forth the contention that mathematics education is mathematical engineering, in the sense that it is the customisation of basic mathematical principles to meet the needs of teachers and students. In next section, we see another model for the understanding of mathematics education before proceeding to a discussion of how mathematicians can contribute to Higher Secondary School Mathematics Education.

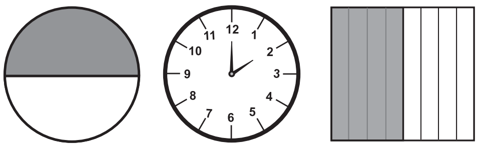
Regarding the nature of mathematics education, Bass (2005) made a similar suggestion that it should be considered a branch of applied mathematics. As mathematical engineering, we emphasise the aspect of engineering to customise scientific principles as per the needs of humanity in contrast with the scientific-application aspect of applied mathematics. Thus, when H. Hertz demonstrated the possibility of broadcasting and receiving electro-magnetic waves, he made a breakthrough in science by making a scientific application of Maxwell's theory. But when G. Marconi makes use of Hertz's discovery to create a radio, Marconi was making a fundamental contribution in electrical engineering, because he had taken the extra step of harnessing an abstract phenomenon to fill human needs. In this sense, what separates mathematics education as mathematical engineering from mathematics education as applied mathematics is the crucial step of customising the mathematics, rather than simply applying it in a straightforward manner to the specific needs of the classroom.

Coming back to mathematics let us see following a practical experience on fraction concepts:

Through this one example of fractions, we get a glimpse of how the principles of mathematical engineering govern the design of a curriculum. The teaching of fractions is spread roughly over classes 2-7. In the early classes, classes 2-4 more or less, students' learning is mainly on **acquiring the vocabulary of fractions** and using it for descriptive purposes. It is only in Classes 6 and up that serious learning of the mathematics of fractions takes place. In those years, students begin to put the isolated bits of information they have acquired into a mathematical framework and learn how to compute extensively with fractions. Fraction concepts develop slowly in some students. **A conceptual understanding** is essential before students become involved in operations with fractions. This time we will see that the **area model of fractions** gives one kind of understanding whereas the **set model offers** another. We will perform an activity to develop both of these important perspectives.

The most basic way of visualising a fraction is part of a whole; this interpretation also is the typical way of introducing fractions to young children.

One person may **"see"** the fraction **"one-half" as a picture of a circle with half shaded.**



This is an example of a **continuous model of a fraction based on area**. The ***area* model** for fractions seems to be the easiest embodiment for students to understand. A critical feature of the area model is that **all the parts into which the whole is divided must have equal area.**

Another individual may **"see"** the fraction "**one-half" as a bag of toffee in which half the pieces are chocolate.** The set model for fractions is more difficult conceptually than the area model (Pyane, Towsley and Huinker 1990). It requires identifying the unit and eliminates the requirement that the pieces be of the same size. Accordingly, it is generally introduced in later grades. This embodiment identifies what fraction of a set has a specific characteristic, such as colour. For example,

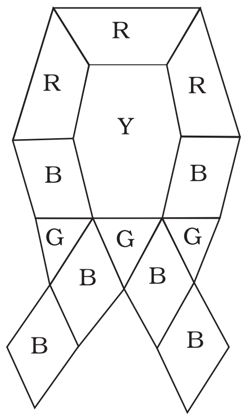
- What fractions of the plants have red flowers?
- What fraction of the people in the room wears glasses?

In this model, the pieces or members of the set do not need to share any attribute other than membership in the set; they do not need the same shape or same area.

It is important to recognise that fractions have different meanings in different contexts.

Through the following activity I shall try to explore the understanding of the area and set models for fractions using pattern blocks and I shall also recommend some strategies that how student should be handled in such type of activities.

Look at the following collection:

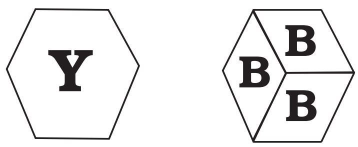


Now the question is "What fraction is blue (B)?"

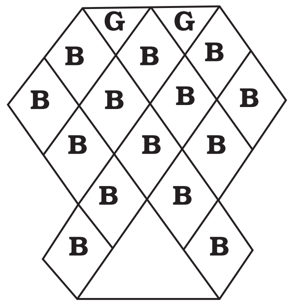
We have responses of several students and teachers as follows:

Responses

1. I think it's 6/13. I think 6/13 is right because there are 13 pieces and 6 of them are blue.
   a. This is the most common response by students.
   b. They seem to think of a fraction as being part of a set, so they count the number of pieces and find that six of the thirteen pieces are blue.
2. I think it's 1/3.
   a. This response is very rare and given by those students seeking additional possible interpretations of the question.
   b. They had simply found the largest piece, the yellow (Y) hexagon block and then had decided that what fraction of the biggest piece is the blue parallelogram piece.

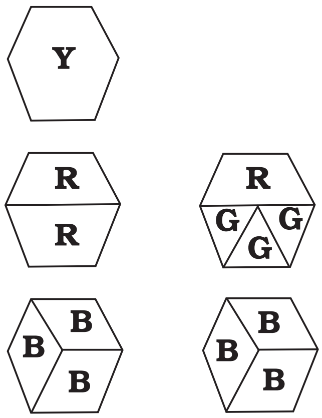


   c. Since three blue parallelogram pieces make up one yellow hexagon block, the blue block must be 1/3 of the largest piece.
   d. This person is using an area model for fractions but is not considering the entire design.
3. It's 1/6
   a. This response is also relatively uncommon but tends to appear more frequently than the previous one.
   b. Here the person has explained that "Design has six blue pieces in all, so one blue piece is one-sixth of the blue pieces.
   c. This student is answering the question "What fraction *of the blue pieces* is one blue piece?"
   d. Without further explanation, it is unclear whether the underlying model being used is one involving area or sets.



4. I think that 6/15 or 2/5 is blue. I think that because I covered completely with 15 blue pieces, 6 of those were really supposed to be blue.
   a. Here student explains that in work with fractions, all the pieces must be of the same size.
   b. Student has usually divided the entire design into pieces of the same size as the blue pattern block. Total blue parallelogram pieces used is fifteen and six of them are really blue.
   c. This one is using an area model, thinking of the fraction of the area that is blue.

5. 2/5
   a. Because if I take all the pieces and move them around so that like colors are together, then how many hexagons will I have?
      (i) One yellow hexagon
      (ii) Two of the red trapezoid pieces
      (iii) One red and three green triangles
      (iv) Two hexagons will be made by six blue parallelograms.



   b. So, I have five hexagons and two of them are blue.
   c. Thus fraction of the design that's blue is 2/5.
   d. For many students/teachers, the 2/5 answer occurs only after they have been encouraged to explore the situation further.

So, finally we felt that all students/teachers seemed to like the idea that different answers could all be correct if each was adequately justified.

Investigating these or similar types of situations can involve students for quite some time. This type of activities is quite helpful to have students write about the activity at two points.

a. By writing before sharing and discussion
b. By writing after discussion, describing what they have learned from the discussion.

For each exploration, students should again have opportunities to communicate about the situation, about their strategies in solving the problem and about their answers. So that students become more aware of their own thinking. The communication aspect of these activities helps teacher to assess each student's reasoning about fractions.

In the primary grades, it is not a serious problem if students' knowledge of fractions is imprecise and informal, so that a fraction can be simultaneously parts-of-a-whole, a ratio, a division, an operator, and a number.

Children at that age are probably not given to doubts about the improbability of an object having so many wondrous attributes. At some stage of their mathematical development however, they will have to make sense of these different 'personalities' of a fraction. It is the transition from intuitive knowledge to a more formal and abstract kind of mathematical knowledge that causes the most learning problems.

This transition usually takes place in grades 5-7. There is by now copious mathematics education research on how to facilitate children's learning of the fraction concept at this critical juncture in order to optimise their ability to use fractions efficiently. At present, what most children get from their classroom instruction on fractions is a fragmented picture of a fraction with all these different 'personalities' lurking around and coming forward seemingly randomly.

What a large part of this research does is to address this fragmentation by emphasising the cognitive connections between these 'personalities'. It does so by helping children construct their intuitive knowledge of the different 'personalities' of a fraction through the use of problems, hands-on activities, and contextual presentations.

This is a good first step, and yet, if we think through students' mathematical needs beyond grade 7, then we may come to the conclusion that establishing cognitive connections does not go far enough. What students need is an unambiguous definition of a fraction which tells them what a fraction really is. They also need to be exposed to direct, mathematical, connections between this definition and the other 'personalities' of a fraction. They have to learn that mathematics is simple and understandable, in the sense that if they can hold onto one clear meaning of a fraction and can reason for themselves, then they can learn all about fractions without ever being surprised by any of these other 'personalities'.

Thus, a coherent mathematical presentation of fractions that provides a logical framework to accommodate all these personalities as part of the mathematical structure is needed. It is hoped that education community will accept the fact that one cannot promote the learning of fractions by addressing only the pedagogical, cognitive or some other learning issues because above all else the mathematical development of the subject must be given careful attention.

From a mathematician's perspective, this scenario of having to develop a concept with multiple interpretations is all too familiar. In college courses, one approaches rational numbers (both positive and negative fractions) either abstractly as the prime field of characteristic zero, or as the field of quotients of the integers. The problem is that neither is suitable for use with sixth class. This fact is recognised by mathematics education researchers, as is the fact that from such a precise and abstract definition of rational numbers, one can prove all the assorted 'personalities' of rational numbers. Since at this stage, we are not able to offer proofs once we are forced to operate without an abstract definition, and that is why we opt for establishing cognitive, rather than mathematical connections among the 'personalities' of rational numbers. The needs of the classroom would seem to be in conflict with the mathematics. At this point, engineering enters. It turns out that, by changing the mathematical landscape entirely and leaving quotient fields and ordered pairs behind, it is possible to teach fractions as mathematics in elementary school, by finding an alternate mathematical route around these abstractions that would be suitable for consumption by children in Classes V-VII.

As of year 2008, the idea is still a novelty in mathematics education that school mathematics can be taught with due attention to the need of precision, the support by logical reasoning for every assertion, the need of clear-cut definition for each concept introduced, and a coherent presentation of concept and skills in the overall context of mathematics.

Of course, there is the old skill-versus-understanding dichotomy, but we also know that such a dichotomy is not what mathematics is about. The conception of a mathematical presentation of fractions is far beyond of partitioning a given geometric figure into parts of equal size only. The need of presenting fractions as a precisely defined concept and explaining each skill logically is not part of these pedagogical picture, lots of story-telling and lots of activities for students to engage in, so that through them students gain experimental and informal knowledge of fractions only. In this way of teaching, informal knowledge replaces mathematical knowledge. A caution of proper balancing is needed. With fractions precise skill with proper intuitive understanding has to be developed. This is an important point that has been traditionally overlooked in education research. One of the main reasons of this lacking is separation of Mathematicians and Educators. Mathematicians generally know mathematics, and educators generally know education. So does it mean that we do not have ample number of 'Mathematical Engineers'?

Let us look, how a mathematical engineer should proceed?

Engineering must mediate between two extremes:

(1) inviolable scientific principles.
(2) user-friendliness of the final product.

What are the inviolable scientific principles in mathematical engineering?

Precision: Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known.

Definitions: Bedrock of the mathematical structure (no definitions, no mathematics).

Reasoning: Lifeblood of mathematics; core of problem solving.

Coherence: Every concept and skill builds on previous knowledge and is part of an unfolding story.

Purposefulness: Mathematics is goal-oriented. It solves specific problems.

What mathematical engineers (i.e., mathematics educators) bring into the school classroom must respect these five basic characteristics of mathematics.

There is no better illustration of this idea of customisation than the teaching of fractions in primary and upper primary classes, we now see, why?

Fractions

No definition. The statement "fractions have multiple representations" is meaningless.

No reasoning. No definition, therefore, no reasoning. E.g. WHY is $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$?

No coherence. "Fractions are such different numbers from whole numbers!"

Poor engineering

Students' failure to learn fractions is well-known. Initially, we get a fraction as parts of a whole, i.e., pieces of a pizza, and this is the most basic conception of a fraction for most elementary students. However, when fractions are applied to everyday situations, then it is clear that there is more to fractions than parts-of-a-whole, e.g., if there are 15 boys and 18 girls in a classroom, then the ratio of boys to girls is the fraction $\frac{15}{18}$, which has nothing to do with cutting up a pizza into 18 equal parts and taking 15.

Thus, a proper engineering for the fraction concept is needed.

The reasons that mathematical engineering is intrinsically bound to both mathematics and education as follows:

1. The customisation of mathematics begins with knowing the classroom needs at each grade level. This requires knowledge of the school mathematics curriculum.

   For example, what third graders need to know about area is different from what tenth graders need to know about the same concept. In addition, even third graders need to know the concept of length before taking up area and they also need to know that the concept of area requires the designation of a unit area.

2. The varied nature of the needs requires the ability to devise more than one correct approach to a given topic. This requires solid content knowledge.

   For example, the meaning of reflection in the plane can be:

   (a) taught by folding papers, or
   (b) defined by using perpendicular bisector of a segment, or
   (c) defined by use of coordinates.
   (d) is appropriate for 5th graders, but not for 10th graders.

3. The nature of the need dictates the choice of the best approach among the alternatives. This requires a deep knowledge of both pedagogy and mathematics: how to reach out to students on their own terms without sacrificing the basic characteristics of mathematics.

It is all too tempting to push aside these basic characteristics in the name of reaching out to students, i.e., it is easy to do defective engineering.

Example: Define $\frac{2}{3}$ $\frac{5}{8}$ to be " $\frac{2}{3}$ of $\frac{5}{8}$ kilograms of sugar", without making precise what it means (what does 'of' mean, and what does sugar have to do with fractions?). This violates precision.

No chemical engineer can function without knowing the fundamental principles of chemistry. No electrical engineer can function without knowing the fundamental principles of electromagnetism. No mathematical engineer can function without knowing the basic characteristics of mathematics.

The idea of customising mathematics "without sacrificing mathematical integrity" is central to mathematical engineering.

The only way to minimise such engineering errors is to have both mathematicians and educators oversee each curricular design. In fact, if we believe in the concept of mathematics education as mathematical engineering, the two communities must work together in all phases of mathematics education. Any education project in mathematics must begin with a sound conception of the mathematics involved and these has to be a clear understanding of what the educational goal is before one can talk about the customisation. In this process, there is little that is purely mathematical or purely educational; almost every step is a mixture of both. Mathematics and education are completely intertwined in mathematical engineering.

Mathematicians cannot contribute to school mathematics education if they are treated as outsiders. They have to work alongside the educators on equal footing in the planning, implementation and evaluation of each project.

There may be some general consequences of a philosophical nature due to isolation. The first one is that the isolation of the education community from mathematicians causes educational discussions to over focus on the purely educational aspect of mathematics education while seemingly always leaving the mathematics untouched. The result is the emergence of a subtle mathematics avoidance syndrome in the educational community. Given the central position of mathematics in mathematical engineering would vanish this syndrome from all discussions in mathematics education?

One other consequence can best be understood as when a system is isolated and allowed to evolve of its own accord; it will inevitably mutate and deviate from the norm. Thus, when school mathematics education will be isolated from mathematicians, so is school mathematics itself, and, sure enough, the latter evolves into something that in large part no longer bears any resemblance to mathematics.

The lack of collaboration between mathematicians and mathematics education may affect professional development as well. The issue of teacher quality is now openly acknowledged and serious discussion of the problem is being to be accepted in mathematics education.

Now as final remarks, it would be mentioned that the concept of mathematics education as mathematical engineering does not suggest the creation of any new tools of solution of the ongoing educational problem. What does it to provide a usable intellectual framework for mathematics education as a discipline, one that clarifies the relationship between the mathematics and the education components, as well as the role of mathematicians in mathematics education? We look forward to a future where mathematics education will act as mathematical engineering, which is a joint effort of mathematicians and educators.

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