Learning from Errors in Numbers and Number Operations in Early School Mathematics

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Abstract

Errors are an inevitable part of the learning process. This article begins with the need to re-conceptualise errors in the learning of mathematics from obstacles or hindrance to insights to learners' thinking process, and opportunities for learning. The latter section of the article focuses on error analysis with reference to the concept of numbers in early school mathematics. It discusses what error analysis means and how it can play an important role in integrating assessment with learning, as well as, help shift focus from right or wrong answers to a broader meaning of learning in mathematics.

KEYWORDS : Error analysis, numbers, number operations, assessment for learning

The National Curriculum Framework (NCF)–2005 bases itself on the principle of mathematics for all. According to this principle, every learner is seen as capable of learning mathematics and all should experience the joy of learning the subject. In conjunction to this, it recommends the assessment to be continuous and comprehensive in nature. Continuous, here, means that assessment should become an ongoing process. The need is to integrate assessment with the daily teaching-learning process, focusing on students' thinking and learning. Comprehensive means to cover a wide range of aspects of learning, like attitudes and skills, (for example, creativity and ability to communicate clearly and analyse) and not simply content knowledge. However, the teaching-learning of mathematics is burdened by approaches focusing on algorithm and one correct answer. Under such an approach, often a learner is evaluated on the basis of his/her ability to get the correct answer. An incorrect response symbolises the lack of understanding. A learner who experiences failure in geting the correct

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answer for some time is vulnerable to be labelled as unintelligent or lacking ability (Boaler, 2013). A plethora of research literature in mathematics education argues for a shift in teaching-learning and assessment practices, focusing more on learners' thinking and responses (Cooper and Dunne, 2000; Lerman and Zevenbergen, 2004; Ryan and William, 2007; Cockburn, 1999). The role of assessment in the development of mindsets and learners' identity is considered crucial. The first section discusses what are errors with reference to number sense in early school mathematics, i.e., Class I and II. The second section explores the scope and need of error analysis as an important tool integrating assessment with learning.

What are Errors?

Learners' alternate responses in given tasks can be classified into two categories. Firstly, like any human failure, learners' alternate response in the given tasks can be a consequence of slips (Ryan and William, 2007). These slips actually have a 'chance element'. These can be termed as 'mistakes'. These slips or 'chance elements' or 'mistakes' do not have any developmental or conceptual explanation. Researchers have found that factors, like misreading and quickly jumping to an answer or conclusions (Sweller, 1994) are reasons for such mistakes or slips. Secondly, the lack of performance can be traced to have a conceptual or developmental basis. A learner's alternate response in a given task can be the consequence of partial, alternate or misconceived conceptual understanding of a mathematical concept (Ryan and William, 2007; Cockburn, 1999). The differentiation between an error and a mistake in a given alternate response is difficult to make. For the convenience of categorisation, if a learner is able to self-correct the response, it can be put in the category of a mistake, else it can be considered as an error. The next section explores the nature and probable reasons for committing errors with relation to number operations.

Errors in Early School Mathematics— An Elaboration through Numbers and Number Operations

The concept of numbers starts developing in children at an early age. Research has brought forth the informal knowledge that learners develop about numbers at an early age (Bryant, 1997; Ginsberg, Choi, Lopez, Netley and Chao-Yuan, 1997). Number concept is one of the core components of school mathematics (Schoenfeld, 2007, and Kilpatrick, 2001). Often numbers are taken as a simple and obvious concept to be learned. But literature describes there are various skills and sub-concepts, which learners may need to learn about numbers with understanding (Cockburn, 1999; Ryan and Williams, 2007). For instance, the simple looking counting process involves pre-number concepts, like one-to-one correspondence, seriation, classification and a knowledge of number names in a correct order (NCERT, 2010). The complexities involved in the learning of numbers and number operations, along with various other factors, like teaching-learning process, language and previous exposure of learners, make errors an inevitable part of the learning process. Errors can be classified on the basis of different criteria. The various sub-concepts involved in numbers and number operations can help categorise the errors learners commit in early school mathematics. For instance, errors in addition and subtraction can be due to the following:

- the lack of understanding of regrouping;
- confusion of 1s and 10s in carrying and writing;
- forgetting to carry 10s and 100s;
- forgetting to regroup when subtracting 10s and 100s;
- regrouping when not required;
- inappropriate use of operation (addition instead of subtraction or vice versa);
- the lack of knowledge of basic number facts;
- the lack of knowledge about the concept of zero;
- over-genralisation: bringing the concept or rule learned for one sub-concept or concept to other where it does not fit; and
- prototyping: generalisation of a concept or sub-concept to only familiar or commonly used examples or situations.

(Adapted from Ryan and Williams, 2007)

With the teaching-learning process in focus, the three major factors contributing to errors in number operations are discussed below. Firstly, teaching or following thumb rules contributes to errors in number operations. By thumb rules, one refers to the shortcuts that teachers tell learners or learners follow in order to arrive at a solution quickly. These thumb rules restrain learners' engagement with a concept, i.e., logic or meaning of the concept. For instance, as we have discussed in the example mentioned in the section on error analysis, there is a possibility that the teacher used thumb rules or the child remembered the rules. In number operations, if on adding two 'ones' digits we get a two digit, then one of the digits needs to be taken to the other place. But the child fails to understand the logic and takes over any of the two digits to the next place.

Secondly, errors can be due to erroneous teaching and learning, i.e., content or unintended aspects. For example, during the teaching of area and perimeter, it was observed that a teacher throughout the unit used cm and km as units of area. And, interestingly, when learners from that class were interviewed on problems related to area, they gave responses for units of area in cm/m/km (Arora, 2011). Here, it can be said that erroneous teaching by the teacher might have caused errors made by the learners.

Thirdly, errors can be due to the usage of examples, which may lead to over-generalisation or prototyping. For instance, in case of number operations, the keyword 'more' is generalised for addition. Consider a problem situation — Fozia sells flowers to passersby on a red light in Delhi. She sells the flowers in bunches of 12. She has nine flowers. How many more flowers does she need to make a bunch of 12? Some children in such a situation may add 9 and 12 and give 21 as their response, instead of subtracting 9 from 12. This can be the case of over-generalisation or prototyping, where the children may have responded due to the usage of the word 'more' in the problem.

The above mentioned classification and reasons for errors are suggestive and not comprehensive or decisive in nature.

What is Error Analysis?

The following work illustrates an error made by a Class II learner in the concept of addition with regrouping.

Here, the child was unable to get the correct answer for some sums. But interestingly, there was a pattern in which the child gave the responses. It would be unfair to consider that the child had no idea of addition. He probably had some idea of addition with single digits, i.e., addition without regrouping. He also knew that if the sum of two digits at 'ones' place resulted in a two-digit number, then one digit had to be taken to the other place. But probably the child did not have an idea of place value. Also, he was unable to reason out which digit should be taken to the next place for regrouping and why? This error analysis is probabilistic in nature, given the lack of evidence.

To reach a certain informed understanding of the child's thinking process, it would be required to give more focused tasks or sums. It is even more important to talk and let the child articulate what he is doing and why. This would be crucial in understanding the problem area and what needs to be done to address it.

Why Error Analysis?

The following learners' responses can be useful for various reasons. Firstly, it can be a useful tool for teachers, who can employ assessment in a continuous and comprehensive manner. Secondly, it can help break the conventional notion of teaching-learning of mathematics, where mastery to reach the correct answer is a dominant practice. It instead can help promote a discursive classroom, where the process of learning becomes as important as mastery over a concept or procedure. In the process, learners are encouraged to think mathematicly by having mathematical discussions, logical arguments and develop an in-depth conceptual and procedural fluency (Ryan and Willams, 2007).

Integrating Assessment with Learning

Grigorenko and Sternberg (1998) argue for a dynamic form of assessment, which does not evaluate learners but focuses on their thinking processes and helps in understanding their current abilities to support the development of their potential. Thus, it is argued that dynamic assessment does not restrict evaluation to the final outcome but gives access to learners' thinking process and potential to learn simultaneously (Lidz, 1987, 1991, cited in Sternberg, 2001; Grigorenko and Sternberg, 1998, Shephard, 2000). Error analysis can be one of the key components in assessment for learning in mathematics (Hodgen and Askew, 2010).

As discussed above, error analysis can help facilitators get an insight into the learners' thinking process and complement assessment for learning. It emphasises on conceptual gaps and turns them into opportunities for teaching and learning. The teaching-learning processes and assessment practices, which utilise errors as opportunities of learning, can help create a positive learning environment. It helps avoid the labeling of learners as poor, weak, or unintelligent in mathematics. Error analysis helps teachers understand what a learner may know and needs to know instead of labeling him/her for what he/she does not know. Teachers may start considering errors as natural steps towards learning. This may help in moving away from labeling learners as intelligent or unintelligent, and provide qualitative feedback, which supports further learning. Qualitative feedback, which is elaborative in nature and focuses on effort and learners' thinking, can help promote the develpoment of growth mindsets among leaders.

Learning in Mathematics: Moving beyond Right or Wrong

A vast literature points to the fear and anxiety many learners associate with school mathematics. It also points out that learners' views about themselves in relation to mathematics is found to be influenced by their marks, ability to give correct responses, and how teachers, peers and parents rated them in the subject (Boaler, Wiliam and Brown, 2000; Reay and Wiliam, 2009). For instance, Boaler, Wiliam and Brown (2000) found that teachers somewhere considered getting to the correct answer quickly without committing mistakes as a marker of one's ability to solve mathematical problems.

On the other hand, an emergent body of literature points out how mistakes should be seen as a stepping stone to learning (Dweck, 2012, cited in Boaler, 2013). Errors are a natural and an inevitable part of learning. Instead of focusing on what the learner does not know, error analysis helps to understand what he/she knows and what he/she needs to know. It helps in designing the teaching-learning processes, which can lead to the development of potential abilities in learners (Hodgen and Askew, 2010). Errors signify the active involvement of learners in the learning process. They help in shifting the focus on the process of learning, learners' effort and thinking process than merely seeing the child's work in terms of correct or incorrect responses. For instance, in teaching a concept, a special session on learners' alternate responses can also become a part of the teaching process. An especially designed worksheet, containing learners' alternate responses, can be given to learners in groups to discuss. They can be asked to decode how and why some child gave such a response, and what was his/her logic. Such error eliciting tasks would help learners gain an in-depth understanding of the concept. Also, it encourages in establishing a motivated learning environment, where the learners are not afraid to make mistakes and see them as an inevitable part of the learning process. This, consequently, can help develop a better self-esteem among learners, positive attitude towards learning mathematics and growth mindset towards learning the subject (Dweck, 2006).

For error analysis to be evidence-based, it is important that multiple sources are used to collect data. Based on the initial data collected, a teacher should use one-to-one task-based interviews to understand the thinking process of the learners behind such responses. These interviews can consist of varied activities, like worksheets, or working with concrete learning material, or oral problem situations. These interactions can be taken as a means to encourage the learners to articulate their thinking and reasoning. This would help find the patterns or logic behind the learners' responses. For instance, the section on error analysis mentioned above attempts to illustrate the initial process of error identification and analysis, using a learner's response to an addition of two-digit sum, requiring regrouping. But this error analysis cannot be considered complete unless the learner is given an opportunity to articulate his/her thinking and the logic behind it.

Tools, like in-depth observations of teaching-learning process, assessment practices, like study of learners' work, one-to-one interviews and focused group tasks can be used to recognise the common errors learners make while learning various concepts in school mathematics. Based on the data collected, the identified errors in a particular concept can be analysed and categorised through thematic analysis. A classroom intervention can be designed based on themes that emerge from the thematic analysis of the errors that learners make. The intervention can be undertaken either with a purposively selected group of learners facing challenges or, generally, with all learners depending on the need and context of the classroom. For instance, in the above case, where the learner was making errors in addition to regrouping, the intervention can focus on both number concept, as well as, place value. Skills, like estimation and checking, can help a child identify the inappropriate procedure he/she is using to add. Varied forms of activities, such as open-ended tasks (Boaler, 2013; Sullivan and Lilburn, 1997), tasks using teaching-learning material, like arrow cards, play money, bead strings, board and card games can help the child build an in-depth understanding of the concept.

Conclusion

As literature suggests, there is a need to pay a greater attention to assessment practices, which contribute to students' learning (Stiggins, 2002; Black, 2004; Shephard, 2000; Ruthven, 1994). However, workable ideas, which can help integrate assessment with learning still remain a challenge. Error analysis can provide a tool kit to design strategies for implementing assessment for learning, in general, and mathematics, in particular. As discussed in the article, error analysis of responses provides a window to learners' thinking and learning process. It helps in understanding what a learner may know and needs to know instead of labeling him/her for what he/she does not know. It may help learners, as well as, teachers to appreciate the incremental nature of intelligence and abilities, along with the development of a growth mindset, where efforts and curiosity to learn and accept challenging tasks become a part of the learning process. But prior to this, there is a need to accept errors as a natural and an inevitable aspect of the learning process. It is only then that errors can be re-conceptualised from hindrance or obstacles to insights for learning (Ryan and William, 2007) and error analysis can become part of teaching-learning and assessment practices.

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