Neha Kashyap
nehakashyap88267@gmail.com

Number Sense and Integers

Abstract

This paper argues that a lot can be understood about the difficulties children have in studying mathematics by looking at their responses to worksheets or other given tasks carefully. It suggests a programme of teaching that focuses on concepts and engagement with abstractions, problem solving, logic and other capabilities in an hierarchical manner as is required by mathematics.

Background

Mathematics as a subject is known for its hierarchical nature and linkages through a structured development of concepts. If we look at the example of place value from primary mathematics, then it can be seen that place value links with addition and subtraction, and if someone has learnt addition it helps in building understanding of multiplication of whole numbers for which it is in a sense, repetitive addition. It can also be extended by understanding division as the inverse of multiplication and for whole numbers also as repetitive subtraction. An ability to deal effectively with place value may indicate ability to deal with whole numbers and operations on them, but then the introduction of negative numbers and fractional numbers poses new challenges. All the rules about the linkages and generalised notions about how numbers behave seem to have been rewritten. This becomes a major stumbling block in the further learning of mathematics.

As we move towards secondary classes, the nature of dealing with numbers changes. From dealing largely with positive counting numbers and operations on them to yield one definite answer, we move on to numbers that are of a different nature such as negative numbers, rational numbers sets and multiple answers. The bringing in of negative numbers that are smaller than zero in the middle school means extending their schema. In addition to this operating on integers and interpreting them as answers involves abstraction. The notions related to these numbers become challenging due to the short cut methods which offer instant solutions. The pace at which they are introduced and the manner of their introduction also make them difficult to understand.

We also deal with numbers in general terms and think about properties that apply to all numbers in a certain categories. The sets of numbers that we deal with become more than odd and even, multiples and factors, etc. and the new sets also cut across the categories that had been created earlier. The high school programme helps us deal with these generalised sets of numbers (whole numbers, negative numbers, integers, fractional numbers, rational numbers and so on) and their properties. Each set has its own defining parameter and from that arises its properties. The work with these sets in generalised form as well as in terms of specific numerical calculations is part of the secondary school programme. These operations also get complex and abstract with the inclusion of literals. As these become more demanding, the struggle gets tougher and it is important to explore the various elements of this. One element in this is the sense of negative numbers and the operations on them.

In this article, we explore some of these gaps in learning and study the way children construct answers to questions around integers and rational numbers and what their answers tell us about their possible understanding.

Introduction

This work is based on the children participating in an intervention programme named 'Shiksha Sambal Programme' for secondary and higher secondary classes. The programme has been effective from year 2016, reaching over five districts of Rajasthan with 60 state government schools. The programme is aimed to facilitate children's understanding in three subjects: Mathematics, English and Science. The important aim is to identify the learning gaps of children and support them at such points so that they can create a path for themselves to clear the exams.

The Children and their Background

These children study in government schools. Most of them come from a low socio-economic background. These children were studying in secondary classes, i.e., 9 and 10 and many of them are first generation learners at secondary level education. These schools have a reasonable teacher- to- student ratio. The responses were collected with the knowledge of the school and the teacher and were a part of the process that was aimed at helping teachers understand children better and also recognise their difficulties. The set of questions in the sheet given to them were at different levels. Some were at the level of the upper primary classes while others were at the level of end of class VIII and IX. While the numbers used for the analysis are small, they are typical of a much larger data we have from all schools and are reflected in our interactions with children in individual conversations or classroom interactions. While we looked at the responses we also had a conversation with a few students about their answers and why they thought their answers were the correct ones.

What We Found...

As mentioned earlier the questions were constructed to help us gauge the level and get an understanding about the way children had constructed their mathematical ideas. We included in this exercise some foundation concepts that we think are essential for learning mathematics upto secondary classes. These include number sense, notion of equality, notion of identity, ratio and proportion, nature and properties of numbers and shapes, notion of letter numbers and equations, understanding data and its analysis, signed numbers and mathematical symbols and language. It has been pointed out by many authors that at the upper primary and secondary stage, some major stumbling blocks are ideas of what are letter numbers, variables, equations, operations with polynomials, understanding of algebraic identities or the reason formulas work, visualisation for geometry and proofs. But much before that integers or signed numbers are a major source of difficulty. They are the first blockage children face as they are contrary to instinct and intuitive understanding of quantity. For us building in children the understanding and capability to work with integers is a key step in moving forward in learning mathematics.

In the following, we present the responses of children to various problems linked to this. We have analysed the answers and tried to see in them a pattern that seems to be able to describe the responses. As said earlier Subsequently we met some of the learners and talked to them about their responses.

In this we analyse only a few of the questions. These questions;

1) Is '-2' a whole number or an integer? Why?
2) Is 3 an integer or a rational number? Why?
3) What is the value of 0-4?

4) What is the value of -6-(-10)?

The questions are addressing different elements of the understanding. They can be said to be at different stages of learning and confidence as well but together they would help the learner and the teacher get a sense of the understanding. Some of the questions like the first and second appear simple, but they have to be responded to with a reason for the answer. For the second question in particular the expectation is that when you give the reason you should be able to recognise that it is both an integer and a rational number. From a class IX student who understands number sets we should expect the answer that it is an integer and hence also a rational number as all integers are rational numbers as well. For the first question there were a variety of responses, from the 22 students of class 9, who were used to doing worksheets with mixed questions. From these, we have picked this question to analyse, as it is a simple one and basic to all that they need to learn in class 9 and later on. There were two students who gave the correct response and also were able to point out that it is an integer as negative numbers are not whole numbers. The rest of the answers can be classified into 3 broad categories. The reason for the response is given alongside.

(a) '-2' is a whole number because whole numbers range from – to + till infinity. A similar response was '-2 is a whole number, whole numbers range from– 1 to + till infinity. Integers are 0, 1,2,3,4........... till infinity. Whole numbers start from – and integers start from 0.

Fourteen students gave this response. The difficulty here seems to be the term for both these in Hindi. The terms are Purnank (Integer) and Purn Sankhya (whole number). Also, it seems the term whole number has somehow got linked to completeness and being divisible. The term पूर्ण संख्या may link to other associations of perfect squares and divisibility.

(b) It is an integer because it is a number which can be completely divided.

Two students gave this response. The sense here again being of completeness and hence of divisibility perhaps coming from complete divisibility. The way this can be understood is the use of the word purnank for integer. This word also indicates in some sense completeness and may have a sense of being composite in some sense.

(c) -2 is an integer. Those numbers that can be fully divided and written in the form of p/q are rational numbers. Those numbers which can't be fully divided and written in the form of p/q are irrational numbers. the response was those which can be fully divided and written in the form of p/q are rational numbers and those that can't be written in the form of p/q are irrational numbers. 0 is a whole number. -2 is an integer.

Five students gave this kind of response, a complex formulation involving all the terms that they have been taught to categorise numbers. There are confusions between all these terms and the manner of defining them and the relationships and results arising from the definitions. The responses are a mixture of all terms linked to what is rational number and what is its form. This response is however, triggered by negative numbers being different from counting numbers.

This second question indicates an interesting pattern also. From the 100 responses, 39 said 3 is an integer, 18 said it is a rational number and while 25 said it is both, the reason was appropriate only in the case of 4 responses. 9 students said it is irrational and 2 said it is irrational since it is integer and rational. Interestingly 7 said it is neither of the two. The reasons for the choices can be categorised in the following way:

(a) Not an integer. Because there are both positive and negative numbers in integer. Therefore, it is a rational number.

(b) It is an integer, and not a rational number because we can't write 3 in the form of p/q.

(c) It is an integer. It is a rational number also because it can be represented on a number line.

(d) 3 is an integer. Not a rational number because 3 is an integer.
(e) No. But 3 is a rational number because it can be written as 3/1.

The responses indicate again a struggle with the number and the definition of their elements. These are attempts to articulate the form p/q, the depiction on the number line and also the signage of the numbers. The attempts however get muddled in bringing out the implication of these for the number 3 and its category.

Operation on Integers

We also looked at responses for operations on integers. For that we analysed question 3 and 4. In question 3, only one third did the question correctly. While the original question did not ask for reason we asked some of the respondents of each category about the reason why they gave the answer they gave. The answers that are not correct can be categorised in the following manner;

(a) 0-4 = 0, because nothing can be subtracted from 0.
(b) It is 4-0 = 4, because we can't subtract a small number from a bigger one. Therefore, we subtract 0 from 4.
(c) 0-4 = 4, because subtracting anything with 0 leaves it unchanged.
(d) It is 10-4 = 6, because we can't subtract a small number from a bigger number therefore borrowing 1 and making it 10 to perform subtraction.

These answers show that the students have tried to construct answers from a framework of understanding that they use to respond to the tasks they get. These frameworks are alternative to the actual framework of understanding for conventional mathematics. Not being able to grapple with the idea of negative numbers most of the explanations attempt to find suitable response from within the understanding of properties and operations on natural numbers. These answers reflect the inability to relate to and accept the notion of negative numbers.

The data from various responses given by 100 students are as follows:

Response	4	-4	0	6	1
Frequency	48	32	14	5	1

In response (a), the child has a narrow understanding of positioning of 0 in number sets. Here it can be seen that there is no relational understanding of negative number 0 and the cognition negative numbers are less than 0. As you keep decreasing numbers from the right of the number line we can cross and then go on the other side and reach negative numbers. And as we move further the numbers become smaller and smaller further to the left.

The responses (b) and (c) are the same but the reasons given are different. In (b) the child is thinking that there is a problem with the expression, and that this can't be solved. Therefore, the recourse is to convert the situation according to his/her own understanding. In response (c) the child is thinking of 0 as the additive or subtractive identity where 0 makes no impact on the other number while operating.

The response (d) is the most striking as the child does not have even a sense of subtraction from a whole number because borrowing is not clear to the child. The rote memorisation of place value is dominating the estimation of numbers without recognising that there is only 0-4 and no place to borrow from.

It is alarming that almost 50% children are giving the wrong responses. Based on the above responses the following can be said;

(i) Children have a limited sense of zero.
(ii) They don't see the existence of negative numbers and that they are smaller than 0.
(iii) Children are following the place value similarly as they do in whole numbers. From where the borrowing of 10 is coming in response (d) is also making the understanding of operating whole numbers questionable.

The fourth question showed the following response pattern:

Response	4	-4	-60	-16	6	-6
Frequency	43	31	12	8	4	2

Less than the half of the children chose +4 as an option and that is interesting in itself. We will not discuss the first as the second is an error in understanding the sign that is important as well but the remaining show the real challenges. We But more interesting is considering the remaining responses and reasons given for the children for these questions when they were asked how they had got their answer. Some of these reasons are;

(a) -6(10) = - 60, because minus into minus is plus. So 10 is multiplied by 6 gives us 60 and because there was a minus with 6, the final answer will give -60. (based on the response in conversation with some of them)

(b) 6 + (-10) = -16, because 6 plus 10 gives us 16. We have minus and plus minus is minus hence -16.

(c) 16 – 10 = 6, because we can't subtract 10 from 6. Therefore, we have to borrow 1 to make it 16 to perform subtraction.

(d)-6 , because we cannot subtract 10 out of 6. Therefore, subtraction can't be performed and –6 will remain as the answer.

Clearly these responses indicate the difficulties with the manner in which students use shortcuts and quick guide clues that are given. The generalisations made by them from the techniques given to them for certain specific contexts lead them to use them in situations where the 'rule' is not valid. For example, for subtraction in the initial classes, the rule given is nothing can be subtracted from zero so borrow from the next column. So if there is no next column what do you do? You borrow from a column which you imagine and hence subtract 10 from 16 or in a similar manner you can say 0–4 is 4 as nothing can be subtracted from 0.

To illustrate one more confusion that is emerging clearly linked to two operations together.

(i) Meaning of rules like —,— is plus implies that when two numbers have — and —, then together it becomes + is suspect as its validity only for multiplication is not stated in the short cut. And the use of brackets is not understood but applied as a memorised BODMAS rule. And that often goes wrong due to mechanical application of it.

(ii) A similar mistake is choosing the addition operation first then, putting in the sign from —,— rule.

Clearly these arise from some of the typical rules for operations on signed numbers

+ + is + and – – is +

– + is – and+ – is –

These four combinations are intended to help the learner understand that multiplying two positive integers' leads to a +ve sign in the answer and a positive and negative multiplication leads to a –ve sign and so on. Another dictum is when the negative and positive numbers are 'together', the sign is of the bigger number. All these get mixed up as the rationale and the reason is not formulated by them. Receiving them as rules to be followed leaves the child in confusion and not only about the number but also about what sign it will retain. The rules given by the teacher suggest treating them as operations on whole numbers and adding the sign afterwards following the rule. Disconnected from the visualisation, quantity sense and understanding of negative numbers, they are recipes for trouble and erroneous generalisations.

To Summarize...

The development of abstraction, logic, perception, generalising, forming hypothesis is important to deal with the understanding of number sets. The children also tend to build their notions about the concepts. The teacher should be aware of these and address them to resolve misconceptions. The unclear understanding of operations on integers keeps on reflecting in the other concepts also such as algebra where the children are supposed to operate polynomials and solve equations.

Therefore, a clear and sustainable approach is needed to strengthen the concepts like negative numbers and their operations. Learning sign rules in isolation to operate integers is not helping children effectively. In this kind of scenario, mathematics will become confusing and torturous for children. This also leads them towards quitting the subject in the higher classes.