**K. Subramaniam**
**ravi.k.subra@gmail.com**

# Giving Meaning to Numbers and Operations in Arithmetic

**Abstract**

This article discusses with examples how basic abstract structures of Mathematics can be taught with examples from real life situations. The author gives examples of contexts that can be used basic for operations of Mathematics.

## Introduction

The subject of Mathematics deals with objects and structures abstracted from general patterns. Numbers and operations on numbers are among the basic abstract structures of Mathematics, but Mathematics also involves the application of these abstract entities and their properties in asking and answering questions about the real world. In fact numbers arise through a process of abstraction, from our actions on the real world such as counting and comparing discrete collections of objects.

In learning about the abstract objects of Mathematics and ways to deal with them, it is best not to start directly with them, but to start with real situations. This is the way Mathematics emerged in History, and is the appropriate approach to give children entry into Mathematics. Many research studies inspired by Piaget and others, have shown the usefulness of starting with real world situations in learning Mathematics. It gives numbers and operations meaning in terms of real or realistic situations and supports the initial learning of abstract objects of Mathematics. Approaches to teaching maths that are prevalent in many schools today may be different from this – they may start directly with abstract entities such as numbers, addition or multiplication and then teach children how to apply these to situations through solving word problems. Such approaches are not as effective as approaches that involve starting from situations, solving context based problems and slowly building an understanding of numbers and operations.

Many teachers believe that in order to solve context based problems children must first be taught how to solve them. For example, young children must first be taught the addition algorithm before they can solve addition problems. Teacher needs to understand that children can and do find their own ways of solving problems. Of course, they may not use the standard method that the teacher has in mind, but children's own ways of solving problems are very powerful starting points for learning eventually more efficient and standard approaches to solving problems.

Let us take an example. Suppose I ask a young child who has learnt counting, how many stones are in my left hand (say 5) and how many are in my right hand (say 3). Then I cup both my hands together with the stones inside. If I now ask the child how many stones there are in my cupped hands, she will find her own way of adding the numbers to answer the question. It will be interesting to a teacher to see how she solves this problem. With insights about how different children approach this problem, the teacher will be better equipped to teach addition. The same context can also be modified into a subtraction problem. Suppose I have a pile of stones on the floor, say 9 stones. I take away some in my hand. The child counts how many are left – say, 6 stones and tries to guess how many are in my hand. Even if the child has not been taught subtraction, she may find her own ways of solving this problem. Similar examples can be formed for children at all ages.

Although the situation described above is very simple, it can be pedagogically effective. Presenting a child with a situation and allowing him or her to find her own way of solving it can lead to successful solutions by students even for problems that are more complex than the example described above. If a division problem is posed in the context of equal sharing for example, 9-10 year olds may find their own ways of solving the problem. For teachers, choosing the right context and framing the right problem can be very powerful pedagogical tools that give meaning to numbers and operations. Different children may respond to different contexts depending on their experience. If a teacher has a well organized example space of contexts and situations, she will be able to adapt them according to the needs of particular groups of students. In this article, I'll try to present frameworks that describe the kinds of contexts that can be used for the basic operations of arithmetic.

## Addition and subtraction

Researchers have suggested that situations involving addition and subtraction are basically of three kinds – combine, change and compare. In the *combine* type of situation, there are two (or more) groups or collections that are either brought together or thought of together. These might be men and women in a group photo, those who are seated and those who are standing, or children and adults. The situation may not involve groups actually coming together , but only thought of together, say the combined population of two adjacent villages, or the total number of motorcycles produced in two different plants of a company. A simple schematic diagram that represents the combine type of situation is the following:

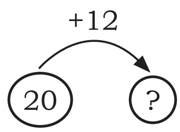
$$(10) + (8) = \boxed{?}$$

The diagram may even be presented to children along with suggested words (example notebooks, textbooks) and they may be asked to form questions. The child, for example, may form the question: There are 10 notebooks and 8 textbooks in my shelf. How many books are there in all? Note that the diagram can be modified in a way that the question is changed , but not the situation.

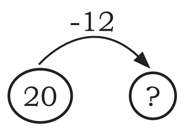
$$(40) + (?) = \boxed{18}$$

In this case, the question would be: There are 10 notebooks and some textbooks in my shelf. Altogether there are 18 books. How many are textbooks? The unknown here is not the sum , but one of the addends. This can be thought of as a subtraction problem. However, the way in which the child actually solves the problem may be similar to addition, for example she may find the answer by counting up from 10 till 18 is reached. The diagram can be modified so that the question mark is placed in the first circle to yield a type of problem known as "start unknown" combine problem, which is known to be more difficult for young children than the unknown addend problem shown above:

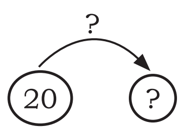
The *change* type of situation involving addition and subtraction involves the *increase* or *decrease* in some quantity or number. Of course, the combine and change type of situations are sometimes difficult to distinguish – this is not a hard and fast distinction. An example of a change situation is “There are 20 people in a bus and 12 more people get in”. This situation can be represented by the following schematic diagram.




If the number was to *decrease* (12 people got down from the bus), then we could represent it using the following diagram.

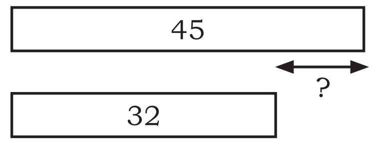



While the problem now looks like a subtraction problem, even the earlier “increase situation” can be changed from an addition into a subtraction problem by changing the position of the question mark as follows:




Indeed, by changing the position of the question mark in each of the diagrams above (there are three possible positions), we can get different problems corresponding to the same context. It is an interesting exercise to think of problems corresponding to each type. We invite you to construct such problems and to try solving these problems with children. Of course, it would be even more interesting to invite children to form their own problems based on the diagrams and the variations.

Finally the compare type of situation can be represented by the following diagram.




A situation that corresponds to the diagram above is: I have 45 story books with me and my friend has 32. How many more books do I have? One could also frame the question as “how many books less (fewer) does my friend have?” Another way to frame it is as an equalization problem: “How many books should my friend get so that we have equal number of books?” For younger children, researchers have reported that equalization problems are easier to solve than comparison problems.

Again, we can see that the diagrams may be modified to place the question mark in different positions to yield different questions. The vocabulary used can also be varied (“more”, “less”, “equal”, “the same as”) to formulate different questions.

**Kinds of numbers**

In the examples that we discussed above, the numbers were whole numbers and the situations involved discrete collections of objects that can be counted. As children grow older, they learn to deal with situations involving not only counting, , but also measurement. With measurement we also move beyond whole numbers. The measurement of common attri, butes such as length, weight, volume, time, monetary value, etc., begins by choosing a unit and producing a measure in terms of multiples of the unit or parts of the unit. When an attri, bute or a part of the attribute is less than the unit, then the unit needs to be subdivided and applied to the part in question. This requires us to go beyond whole numbers to fractions (or positive rational numbers). Thus the situations of combine, change and compare can also involve quantities that are continuous measures and not just discrete quantities. However, the same framework applies without much change.

A second jump occurs a little after children have been exposed to fractions – that is the introduction of negative numbers. One of the difficulties that children face is in interpreting negative numbers. What does "–2" exactly mean? There are three broad senses in which negative numbers, or more generally signed numbers (positive, negative numbers and zero), are interpreted in situations.

1. **As a state**: We can specify the state of something we are interested in using signed numbers, but only when it is meaningful to talk about positive and negative states. Some examples are temperature of water in a freezer, height above and below sea level. (You can try to think of more examples.)

2. **As a change**: Signed numbers can denote change along with the direction of change: increase or decrease, movement up or down (or forward and backward) or positive or negative growth (for example total annual sales of a company). An interesting example that some teachers suggested in a workshop was to make a table of the weight gained (i.e., change in weight) by a baby every week. (What does a negative change in weight indicate?)

3. **As relation between numbers or between quantities**: An example that illustrates this meaning is the following: Think of a pilot in an aeroplane circling near an airport with many other planes in the air, all waiting to land. The pilot would be interested in knowing the relative height (or altitude) of the closest plane with respect to his own plane. The relative height may be positive (indicating that the closest plane is above the pilot's plane) or negative (indicating that the closest plane is below the pilot's plane).

A more detailed discussion of the kinds of situations in which signed numbers can be applied is possible and very investing, , but it is beyond the scope of this article. We will just note that it is possible to think of addition and subtraction problems involving signed numbers in ways similar to whole number contexts discussed above. That is, we can have combine, change and compare type of situations for signed numbers, in which the signed numbers themselves take on different meanings of state, change and relation. Some combinations of situations are more natural and some are more contrived (Vergnaud, 1982). We will leave this again as something to be explored by the reader.

## Multiplication

When we think of situations where numbers are multiplied, a striking difference from addition emerges. When we add two quantities, these quantities are of the same kind and in the same units: we can add two lengths in meters or two amounts of money in rupees, etc. We cannot, of course, add 5 meters and 10 centimeters to get 15 metre-centimetre – before adding we must convert one of them into the same unit as the other. However, when we multiply two numbers, it is very rarely that they are quantities of the same kind.

The most common kind of situation involving multiplication can be represented by the following schema:

$$\textit{Rate} \times \textit{Quantity 1} = \textit{Quantity 2}$$

To take an example if a kg of potato costs Rs 25, we can find the cost of 5 kg using multiplication:

Rs 25 per kg × 5 kg = Rs 125

Note that in this example all the three quantities are of different kinds and in different units. The first is an amount of money per unit weight, that is, a rate, the second is weight and the third is amount of money. Most multiplication situations fit this pattern. You can try to make up situations where the quantities involved are length, time, volume, etc. and are combined in various ways. Two other types of situations involving multiplication are related to the rate type of situation, , but are slightly different. These are situations where quantities are scaled up or down by a scaling factor (for example in maps), and unit conversion problems (how much is 2.3 m in cm?). The scaling factor or the unit conversion factor is similar to a rate, , but involves only one kind of attribute or measure unlike a rate, which involves two kinds of attributes or measures.

There are situations where two quantities of the same kind are multiplied, but these are relatively fewer than the situations described above. In fact, in elementary classes the only example is the multiplication of two length measures to obtain area, or the multiplication of three length measures to obtain volume. (Note that, in contrast to multiplication, if we add two lengths, we only obtain another length.) Another uncommon kind of situation is where two quantities are multiplied, where neither is a rate. This occurs in school Physics – in the case of a lever or a balance, we multiply length and weight to find the moment about a fulcrum or pivot.

The situations described above involve measures that are mostly continuous measures. Situations involving multiplication of only whole numbers are even simpler. They usually involve finding the total quantity of equal sized collections of discrete objects, like 12 boxes with 10 eggs each. There is an interesting type of situation of multiplication of whole numbers that is different from these. It involves finding what is sometimes called the cartesian product of two sets. For example, if I have three shirts and four trousers, then how many different combinations of shirt and trouser can I wear?

A final remark is on the question of which types of multiplication problems are more difficult for children to solve. This does not always have an easy answer like situations of type A are easier than situations of type B. Of course, some situations such as equal groups of discrete objects are simpler because they can be modelled by children using icons, objects or even mental objects. However there are many factors that make a problem relatively easier or more difficult – the familiarity of the situation, the language and vocabulary in which the problems are posed, how big the numbers are, what type of numbers they are, the relation between the numbers, etc. As a teacher works with particular groups of students, by varying the situations, a teacher will develop an understanding of which problems are easier and which are more difficult. Even better, teachers could form a group and try out different variations of a problem and share and discuss their findings with each other.

## Division

Division is the inverse operation of multiplication. So division situations are related to multiplication situations. However since multiplication commonly involves two kinds of quantities that are multiplied, division can be interpreted in two ways. Let us consider first only multiplication of numbers

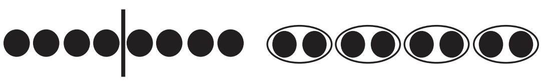
25 × 5 = 125

We can have two division facts that are the inverse of this multiplication:

$$125 \div 5 = 25 \text{ and } 125 \div 25 = 5$$

Let us interpret this now in terms of the situation that we considered above – the cost of 5 kg of potatoes. We could have two kinds of division problems corresponding to this situation: (a) If the cost of 5 kg of potato is Rs 125 than what is the cost per kg of potato and (b) If the cost per kg of potato is Rs 25, then how many kg can I buy with Rs 125? We see that in the first case division is used to find the rate or per unit cost and in the second case division is used to find the number of units.

Even in the case of division involving only whole numbers representing discrete quantities, we see that there are two meanings of division that are similar to the two meanings of division above. Let us take the example of $8 \div 2$. (Note that this is a different starting point from the previous example where we started with one multiplication fact and two corresponding division facts. Now we start with only one division fact and discuss two different meanings.) We can interpret $8 \div 2$ in two ways.



On the left the division shows the number per group when the number of groups is given to be 2. On the right the division shows the number of groups when the number per group is given to be 2. The situation on the left corresponds to the “equal partitioning” meaning and the situation on the right to the “equal grouping” meaning. The equal partitioning meaning corresponds to finding the quantity per unit in the measure context, and the equal grouping meaning to finding the number of units given the quantity per unit.

The remarks above may seem to be rather trivial and obvious. However, knowing the different meanings of division becomes useful when dealing with a question like the following.

- Construct a word problem that corresponds to the operation $1\frac{3}{4} \div \frac{1}{2}$ .

This is a famous problem that was given by the researcher Liping Ma to Mathematics teachers in the USA and in China (Ma, 1999). She found interestingly that almost all teachers from the USA found it very difficult to construct a word problem corresponding to the given division fact. In fact, many teachers suggesting taking $1\frac{3}{4}$ pizzas and sharing it among two people, which corresponds to the operation $1\frac{3}{4} \div 2$ and not to $1\frac{3}{4} \div \frac{1}{2}$. Many of the Chinese teachers, in striking contrast, could come up with several examples of situations. In fact the situations corresponded to three different meanings of division as seen in the examples below:

1. If a machine can lay ½ km of road in one day then how many days will it take to lay 1¾ km of road. (This corresponds to finding the number of units given the quantity per unit, similar to the equal grouping meaning for whole numbers)

1. A wealthy man is partitioning his farm to distri, bute it among family members. Different family members get different shares. If half a share corresponds to 1¾ of an acre, then what is the size of one share. (This corresponds to finding the quantity per unit, given the number of units, which in this case is half a unit.)

1. If the area of a rectangle is 1¾ units, and its length is ½ unit, what is its breadth? (We have not discussed this meaning in the context of division, , but we have discussed it in the context of multiplication. It corresponds

to the situation of product of measures – length × length = area.)

## Conclusion

We have discussed the different kinds of situations that correspond to the arithmetic operations of elementary school and tried to evolve a framework or a categorization of such situations. How is this useful? Firstly, situations give meaning to numbers and to operations. This makes it easier to learn and deal with abstract entities like numbers and operations. Teachers need to have the capacity to design situations and flexibly adapt them to their classroom teaching. I hope that having a synoptic overview of kinds of situations will help in designing appropriate and powerful contexts for learning arithmetic.

When one moves beyond whole numbers to fractions and integers, the range of situations and meanings expands. It is important as well as challenging to design appropriate situations that involve operations with fractions or signed numbers. How does one design a context which is modelled by the multiplication fact $(-3)\times(-5) = +15$? A discussion of this challenge will need one to delve deeper into the meaning of integers, which we will not be able to do here. Some discussion of this and related issues concerning integers can be found in Kumar, Subramaniam and Naik (2015). Similarly, fractions also have different meanings (sometimes called subconstructs) in different contexts (Naik and Subramaniam, 2008).

Further, developing an understanding of how numbers connect with real situations makes one sensitive to what is an appropriate use of number and operation and what is not. This can be illustrated with the help of a puzzle that was recently circulating on social media. Here is the puzzle:

I had Rs 50 and I went shopping. Here is what I spent:

| **Spent (Rs)** | **Balance (Rs)** |
|---|---|
| 20 | 30 |
| 15 | 15 |
| 9 | 6 |
| 6 | 0 |
| **50** | **51** |

The puzzle then asks: where did the extra Rs 1 (in Rs 51) come from?

The resolution of the puzzle consists in realizing that while it makes sense to add the numbers in the left column, it does not make sense to add the numbers in the right column. The numbers in the left column are all amounts of money which are distinct, non-overlapping parts of Rs 50 and together make up Rs 50. The numbers in the right column and not distinct parts, , but parts of Rs 50 which are contained in other parts and cannot be added to represent a whole (i.e., Rs 50). For example if I spent only Rs 1 for the first two items that I bought, the first two rows in the right column would be Rs 49 and Rs 48. If you add them it is much greater than Rs 50! This is because Rs 48 is already contained in the earlier balance of Rs 49. In this example, it is rather easy to find out what has gone wrong and why it is inappropriate to use addition. There are many subtle ways of fooling people using numbers. One of the goals of Mathematics education is to be able to see through such incorrect uses of Mathematics.

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