Richa Goswami
goswami84@gmail.com

Study of Children's Errors: A Window to the Process of Teaching and Learning of Mathematics

Abstract

Errors that children make are often seen only as flaws in learning and rarely as windows to their thought process. Two ways of looking at errors include; viewing them as learning stages and viewing them as gaps in learning. These views become even more pronounced in the teaching-learning process of a subject like Mathematics where the binary of correct answer and incorrect answer is seen to be clearly distinct. In this paper a study of children's errors has been undertaken to understand what they indicate about the current knowledge of the learners. So, the new perspective being proposed in the paper is to view errors as an important resource for the teacher that would help them plan future teaching.

Introduction

Mistakes made by children elicit very different responses from adults based on the context in which they occur. The response and interaction is different if the mistake occurs among peers, in front of elders at home or in a formal setting like school. Usually at home and among family members, the common 'mistakes' of young children are taken very lightly and would elicit supportive, positive and affectionate responses. On the other hand in school, mistakes are generally perceived as undesirable, and are in the category of 'must be eradicated as soon as possible' form of behaviour, but we merely need to observe a child trying to learn something to realize that errors are an integral part of the learning process. This is true for all walks of life, whether academic or non-academic. Let us take a common example of a child learning to use a spoon for the first time. At first, she spills the food, but gradually, after a number of failures, succeeds. Similarly, the mistakes made by a child learning the names of the colours, which she may learn over a period of many months, reveal the systematic errors in the process of learning. At the first stage my two and a half year old daughter used the names of the colours merely as nouns, with little knowledge of what is green. She would point out to objects and say 'this is green'. Then she started using the words in the specific context of colours,, but did not know which colour was what and the third stage which is her current stage, she recognises black, but mixes up the names of all the other colours. Though she is able to identify two things of same colour and take notice of their common characteristic. At each stage of learning about colours, she makes diferent mistakes and as parents we are only focused on what she knows, happily ignoring what she doesn't know.

This paper is based on a study to understand the mistakes made by students and what they reveal about the current conceptual understanding. Place value, a fundamental concept of Mathematics curriculum of elementary classes, was taken as the area of enquiry to understand the nature of errors made by students and the reasons for them.

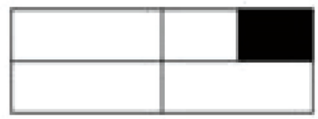
In India, the board to which the school is affiliated to, impacts education in many ways. The specifics of the syllabus, the textbooks prescribed and the assessment procedure depend on the board. Thus, to ensure variety in the data, the sample included children from different settings: two Central Board of Secondary Education (CBSE) schools in Delhi, one CBSE school in Rajasthan, one state board school in Delhi and one state board school in Rajasthan.

Understanding Errors

When we examine the work done by children there are many ways in which we respond to it. When a two three year old child, drawing on a paper with a crayon says I am drawing a grape and then draws a big circle two types of responses are possible, "Oh great this is such a nice grape" without bothering with the lack of likeness. Another response could be, "this is good, but don't you think this is too big for a grape?" These responses indicate two very distinct and in a sense opposite ways of understanding errors. These two ways are:

- **Errors as learning stages:** The errors are a part of the learning process and apart from oversights that the child would not repeat, there are certain errors that would be made by almost all children going through that process. These are unavoidable as these are stages of learning. For example, all children pass through the following stages when learning to speak: Crying, Babbling, One Word/ Holophrases, Two Word Phrases and then Multi-Word Phrases (Aitchison, 1998). Whatever attempts a parent may make to avoid these in their children, these stages are inevitable and are generally not affected by correction except when the favourable time arises.

- **Errors as gaps in learning:** This would imply that children make errors because they have not been told the correct method or have not grasped the correct facts. The child tries to fill these gaps in her own logical, but not necessarily correct way which leads to errors. For example, if a child is shown a rectangle divided into five unequal parts as shown below and she answers that the fraction of the shaded region is one-fifth then the child is unaware or currently ignoring the fact that the five parts have to be equal for it to be one-fifth.



It can be seen that these two views are fundamentally different from each other. In the first view, errors are seen as necessary stages in the process of learning and these reflect the way a child thinks. They are evidences of children's knowledge rather than of their ignorance. And in the second approach they are the evidence of lack of knowledge.

Error analysis helps teachers in understanding what an error reflects about the child's current knowledge status. That is, it allows teachers to diagnose the level of learning of the students. This understanding helps her in modifying her approach to suit the needs of the children. Wrong answers given by students often tell us more about their present and unique state of understanding as compared to their right answers. There is often only one correct answer, but a variety of incorrect ones. Therefore, why and how a child reached a different answer becomes an interesting area of inquiry. Asking the student 'how' the question was solved (or 'speculating' about it based on the teacher's knowledge of what all the child already knows) would reveal a lot about the conceptual structure of that particular child's thought process. We would later talk more about these processes of understanding the errors made by children.

Context of Learning Maths

While learning Mathematics, children are supposed to learn abstract concepts and relationships as well as algorithms and facts (like number facts and multiplication tables). There are three basic elements of any mathematical concept *(Richard Garlikov, The Concept and Teaching of Place-Value, 2000)*. The three elements are:

1. Convention
2. Algorithmic manipulations
3. Logical/conceptual relationships

However, due to over emphasis on learning and teaching of algorithms no time is given to building logical relationships. Teachers have themselves studied through this culture where convention and algorithm is all important. Thus as teachers they by default emphasise the same.

Therefore, children also either fail to grasp the concepts and principles that underlie procedures or they grasp relevant concepts and principles, but cannot **connect** them to the procedures. Either way, children who lack complete conceptual understanding, frequently generate systematic patterns of errors (Siegler, R.S. 2003). Siegler talks of how, depending on the manner in which one looks at it, these systematic errors can either be a problem or an opportunity. They are a problem as they indicate that children do not know what we have tried to teach them. On the other hand, they are an opportunity as they indicate the specific misunderstanding developed by a learner and thus can be worked upon.

Place Value in Mathematics: The way we normally record numbers is known as a decimal 'place value' system (Dickson, et.al. 1984). It is a system of successive groupings where units are grouped into tens; tens are grouped into hundreds, hundreds into thousands and so on.

Place value is one area with which the students start working (intuitively) at a very early stage. This means that they pick up the pattern in which numbers are generated or some understanding that 21 will be followed by 22 and 23 similarly 31 will be followed by 32 and 33. This develeops as children orally learn the number names and start learning to write the number names. When children start writing numbers they pick the number pattern and are able to predict next numbers. The ability to predict numbers is an indicator that they have understood something about the number structure. This is supported by the nature of number names. In English number names, after twenty are regular and indicate their decomposition i.e. Fifty one is fifty and one. (Nunes and Bryant, 1996).

But this is also an area in which students often make mistakes. Place value forms the basis of arithmetic and is thus related to errors in various other topics. As a result, children often form incorrect procedures and inefficient strategies for solving multi-digit arithmetic problems.

Elementary school teachers generally understand enough about how to use 'place value' to teach most students to eventually be able to work with it; but they don't often understand place value sufficiently to help them understand it very well, conceptually and logically. And they may even unknowingly impede learning by confusing children; for example, trying to make arbitrary conventions or giving recipes and short cuts as logical steps. In many primary schools, children chant one one eleven, one two twelve.....and so on, or they write in their notebooks following the two steps given below.

1st Step	2nd Step
1	11
1	12
1	13
1	14
1	15
1	16
1	17
1	18
1	19
2	20

This way of speaking and writing hinders the understanding of number sense.

Stages of development of place value: *As* discussed above learning is an ongoing process. One keeps adding more nuances to the understanding of a particular concept and build more relationships among the concepts. Understanding of place value similarly develops gradually. Ginsburg identifies three stages in developing an understanding of the theory of place value, where the written symbolization of the number is concerned.

- The first stage is where the child writes a number correctly with no idea as to why it is written in this manner. For example thirteen is written as '13' and there is no reason for it.
- The second stage is where the child realizes that other ways of writing a particular number are wrong - for example '31' is incorrect for 'thirteen.'
- In the third stage the child is able to relate the written notation of numbers to the understanding of place value. For example, Doug, a 7 year old, when asked why he had written a '1' followed by a '3' to indicate 'thirteen' replied that the '1' stands for ten and '3' stands for 3. Ten and three is thirteen.

Thus even though ch1ildren in stage 1 and 2, start unearthing patterns and develop an understanding that numbers proceed in a systematic manner, they are unable to articulate that system. Thus, the shaky concepts that they have, do not support in forming effective arithmetic strategies. It would thus be worthwhile for a teacher to understand the depth of learning of the children to take them to the next level.

Sample selection

The sample of the study included children studying in class 7 from five different schools. These schools catered to populations from varied socio-economic background and using different Mathematics textbooks. Three schools used NCERT books, one school used the Delhi state board textbooks and one school had Rajasthan state board textbooks.

Place value is formally introduced in the textbooks of class 3. It is a fundamental concept in Mathematics and later number system, arithmetic and algorithms develop on it. The spiralling of curriculum and working with and on the concept of place value provides students with an opportunity to understand the concpt better. Thus it was decided to undertake the study with class 7th students and the problems for the study were selected from the textbooks of class 6th.

All the students of class 7 were given a test paper. Based on the errors made by students in the paper, three students in each school were selected and follow up work done with them. They were asked to do a few of the questions again along with unstructured interview. While selecting questions to be given to each child in the follow up session, it was considered that the first question to be given to them would be the one which the child had correctly answered in the test paper. Four more questions were given which the child had answered incorrectly. Numbers in the questions given in the follow up were changed. The interview focused on asking the children what they understood by the question, how they think it needed to be solved and then allowing them to solve it while encouraging them to articulate the reason for what they were doing. The interviewer recorded both the strategy undertaken by the child and her articulation of the question and required solution.

Problems related to number sense

Based on the responses in the paper and in the interview, errors have been categorised as follows:

1. Errors originating from difficulty in comprehending mathematical language
2. Errors originating from the understanding of place value

Comprehension of mathematical language: Mathematical language or the language used to communicate mathematical ideas has many peculiar features to it like use of symbols, words having meaning different to the customary ones, use of charts, tables and graphs, language of word problems, use of conditions, etc. This causes an additional challenge for the learners of Mathematics. Like the learning of any other language, the learning of mathematical language is impacted by the amount of exposure and the usage of language to communicate ideas and not focus on the language itself.

Errors originating from difficulty in comprehending mathematical language: There were a large number of mistakes in the paper that were related to the understanding of mathematical language. These mistakes related to use of symbols, understanding the language of the question and difficulty in dealing with questions that had more than one condition.

In the following paragraphs, a brief description of such errors has been presented:

Confusion about symbols - what should be written where, knowing what a symbol stood for, respecting what it means and how we should relate it to the question in view of it was missing in many students. There were several examples where students incorrectly and interchangeably used the comma or the equality sign. For example while writing the number 9857 a child had written 9,8,5,7. In another place an equality sign was used to separate different numbers (7744 = 7474 = 4477 = 4747).

Another error in the use of the comma was that many children in the entire test, did not place the comma even at the places it was needed. Thus to distinguish one number from another space was left. *For example when asked to write the greatest and smallest 4 digit number using all different digits, the child wrote 9876 1023, with no comma to separate the numbers.*

Errors due to language confusion: Another problem which one often faces while trying to assess a child's mathematical learning is how to differentiate between whether the child is unable to capture the concept or is confused with the vocabulary/language.

The first question asked children to write the greatest and the smallest number from a given set of numbers. While answering it, three students wrote the given numbers in ascending and descending orders instead of choosing the greatest and the smallest number. Nine students picked the two greater numbers and wrote them both instead of the one greatest number. When a child was asked what he understood from the question, he said, "We have to set these numbers according to the greatest and the smallest, first we will write the greatest number, then smaller than that and then even smaller. Greatest means from bigger to smaller."

Similarly in the question where certain number names were to be written as numerals, many students wrote the number correctly using Roman numerals. This could be because of the interference of the word 'numeral' that they had encountered in the context of Roman numerals.

Commonly used vocabulary also posed problems for the students in some cases. Some words, when used in mathematical context have a different meaning than in their everyday usage. These include words such as point, equality, chance, etc. One such word, used in the test, was difference. In the question where the students were asked the difference between the two place values of 2 (Q9), 27 students took the word difference in the literal sense. Thus they gave some very interesting answers like 20000 is greater than 20 or it has more number of digits, it has 4 zeroes, etc.

In Q10, where six digits 4, 5, 6, 7, 0 and 8 were given and students were asked to make 5 six-digit numbers and then arrange them in ascending order, some students instead of placing the numbers in ascending order placed them in descending order thus showing that while aware of the ordering of the numbers they are still confused about the nomenclature.

Sometimes when part of the question is done incorrectly and we try to identify the reason for it, we may find that the problem was not with the concept, but incorrect or partial comprehension of the question. In such a case rephrasing the question may help them to find the correct answer. In a question where 4 digits 2, 4, 7 and 8 were given and students were asked whether the smallest 4 digit number (made using the given digits) would be greater than 3000, as many as 30 students wrote the answer 4278. A probable reason for this could be that students misunderstood the question and interpreted it as asking them to write the smallest number which is greater than 3000. Some other answers like 3001 and 3002 also support this understanding.

In lengthy questions, chances of misinterpretations are even higher. In a question "you have the following digits 4, 5, 6, 7, 0 and 8. Using these, make five different numbers with six digits each. Now arrange these numbers in ascending order." 12 students did not make the numbers using the given digits; instead they arranged the digits themselves in ascending order.

In Q3, where the students are asked to expand the question, some students wrote in tenth and hundredth. This is something that these students have started learning in class 7 and thus this probably is interfering with the concepts learnt earlier. In the question asking them to give place value of digits placed at different positions in the numeral for a number, some students said that the place value of 2 remains 2 in any positions. Face value is introduced as a term to students much later than place value; thus a more recently learnt concept is fresh in their minds and also interferes with the earlier concepts. We can say that they are using the more recently learnt concepts and that is because of two reasons. One is linked to the fact that since they are in the process of acquiring these concepts they end up attempting to link it to everything they come across to explore and test if it can be related or not. The second reason is because of the way Mathematics and other subjects are taught where once an idea is introduced the classroom works with those ideas for many days continuously till the next idea comes. So they are all expecting to be given questions related to recently learnt concepts only and hence the wrong interpretations.

In many cases, it was felt that the students could not focus on all the conditions given in a question and thus simplified or solved part of the question. This was a pattern seen across the schools.

In the question where the students had to make the smallest and the greatest 4-digit number using digits that are all different, they simplified the question for themselves by only focusing on the condition of making the smallest and the greatest number. Thus we got to see answers like 1000, 1111 and 9999.

In the other question the task was similar, but there was a slight difference. Here they had to make the smallest 4-digit number while using the given three digits only. In this they were required to use one of the digits twice and could choose whichever they wanted, but most of the students could not perhaps understand and did not comply with the conditions. Many made 4-digit number using any random digit other than the 3 that were given. Some other students used all the given digits and made the smallest 3-digit number possible. Another child gave 1000 as the answer thus focusing only on that part of the question which said the smallest 4-digit number.

Errors related to place value

- ***Using Zero***: Zero has been a problem spot for many children and the inability of students to work with numbers containing the digit zero was seen in many questions. In the question where students had to expand the number 20085 (Q3), many students gave the answer 20000+0000+000+80+5 or 20000+0+0+80+5.

  Thus the students either do not know or are not very confidently aware of the fact that zero is only a place holder and thus has no value that needs to be separately written. Leeb Lundberg (1977) describes some of her problems as a teacher when dealing with zero. Its role as a placeholder, in the symbolic representation of number, is something not readily appreciated by children.

- When asked to write ten thousand and nineteen and thirty three thousand and thirty three as numerals a girl wrote them as 1019 and 3333. When she was asked to explain, she said, “We were asked to write ten thousand and nineteen in numerals, so we write 10 and since zero has no value, so we write 19 after that. Similarly we write 33 and then 33. If we write 3300033 then it will be entirely wrong.” (The response has been translated)

  As we see here students were expected to write the given number names in numeral form, the specific requirement of the question was to use zero as a place holder. Most of the errors that were seen in this question showed that students find it difficult to place zero as a place holder and even when placing zeroes are not sure how many zeroes are exactly needed. Responses like 1000019 and 3300033 are interesting as in these first ten then three zeroes are written for thousand and then nineteen is added. The response is similar in the second case.

- ***Difficulty in working with large numbers:*** There were many examples when it was felt that the problem that students faced was not in understanding of the concept, but in handling large numbers. This is an area where even secondary school children show a definite weakness. Many seem to be unfamiliar with the place names of digits to the left of thousands position (Dickson, L. et.al; 1984). This could also be one possible reason for such a large number of students finding it difficult to write ten thousand and nineteen and thirty three thousand and thirty three. Answers which had many more zeroes than needed could also be because of this. Their competence in dealing with large numbers may not have yet developed enough to check the numeral they have written and what number it actually represents. There were other places where errors indicated difficulty in handling large numbers.

While writing 20085 in expanded form, many decreased or increased number of zeroes, but read out the number correctly. This suggests that if the number of digits are large, students are not able to make sense of the number and get confused. This is what perhaps makes them give wrong answers.

Similarly in the question where the students had to write numerals for given number names, there were several incorrect attempts and again the most frequent mistake was writing a 4 digit number instead of 5 digits.

Many students wrote 2000 for the place value of 2 in 426328. This also seems to indicate that they find the number represented by the numeral too big to handle. It is possible that it is only a slip, but the frequency of this error makes this being only a chance error or a slip a little improbable.

Large numbers seemed to be creating a problem even while carrying out familiar algorithm operations. The operations needed in this test were subtraction, multiplication and division. Though children faced difficulties in all the operations, it was more so in the case of division. While working on division questions, 11 students started working on the algorithm correctly, but left it midway, even though many of them had done it correctly till that point. Thus not knowing how to do the sum was not the reason for their leaving the question; instead it seems that the length of the question scared or bored them. Some kept on doing it, but forgot to write the quotient.

- ***Basic algorithms*:** Many students in spite of being aware of the demands of the question and being aware of the concepts end up making mistakes in carrying out simple operations. Many such cases were seen in the test also. Students who could give the correct place values and had also interpreted the word "difference" correctly, i.e. as per the question made mistakes while subtracting.

  Eleven students while multiplying 1098 with 25, made an error in vertical addition. Eight students forgot to add the carry over. Similarly while working on division questions many students did the subtraction part wrong. Most of the children who make such a mistake when trying to spot it on their own, were able to find it.

  Question 11 required students to subtract 6980 from 10000. In this question the most frequent incorrect answers were 4120 and 2120. Children borrowed from 10 repeatedly, thus 10 minus 8 gives 2 and 10 minus 9 gives 1. From the first 10 either they subtracted 6 to get 4 or considered it to be 8 as twice they have taken one from it thus 8 minus 6 is 2. This indicates how the number notation, relationship between different places and the related idea of borrowing is not clear to them. It also raises questions whether borrowing is a correct idea to be presented with the subtraction algorithm.

Conclusion

The analysis shows the different areas of difficulty students face in working on place value related problems. The challenges include the use of zero as place holder and the algorithm of carry over, borrowing, multiplication and division originating from the understanding of value attached with the place of a digit in a base ten number system.

This small indepth study of the understanding of place value is indicative of the variations and layers that can be seen in the errors made by children relating to a small curricular area in Mathematics namely place value. This has many implications for the teaching/learning process in elementary classes.

The first and perhaps one of the most important implication is about how we view the errors made by children. Instead of the popular view of errors as a sinful deviation which needs instant reprimand and correction, it is possible to view them as a window to the understanding of children. A teacher can form 'reasoned speculations' based on her knowledge of child's learning level and her own conceptual clarity. Collaborative engagement of the learner and the teacher on these errors may lead to learner's development towards correct understanding on her own.

One on one interview often used by researchers to gain deeper understanding of all kinds of issues in various disciplines, is also an important pedagogic and assessment tool. The traditional paper pencil test can only tell us about the question that the child could do correctly and the ones that she could not. One on one interaction on the other hand helps teacher understand the level of understanding as development of a concept is a graded process and not a binary of knowing and not knowing. Asking the child, what is it she understands by the question and what does she think is needed to be done are important mathematical learnings even when she is unable to work the algorithm and makes mistakes in it. The second step is to understand how she would work the algorithm and if she can articulate why this is being done. This articulation on one hand would help the teacher, but on the other hand, it would provide an invaluable opportunity to the learner to reflect on what she is doing, gain confidence about it and understand that Mathematics is not a bunch of baseless algorithms, but an intricate network of concepts.

The question of how much does a teacher need to know to be able to teach is a well debated question. It is linked to the mandatory qualifications required for teachers and the content of the in service programme. This study throws some light on this aspect. Considering the real understanding of concepts in Mathematics has many layers to them, a teacher is on one hand required to know the concept fully and also be aware of the stages in development of the concept. The knowledge of how a concept develops is essential for designing both pedagogic and assessment activities.

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