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Transitions in School Mathematics

The movement from 'real world' problems to formalization to mathematical *Discovery includes many transitions.* (V.I. Arnold). `

An Exploration

What follows is an account of an adult education class, I was involved in, during 1991 in one of the slums of what was then Madras city. The tenements did not have piped water, there were pumps every 100 metres (or so) from which water was collected. In addition, tankers would arrive periodically bringing water, and residents stored them in buckets (and containers called *kudams*).

Many learners were often late to class, and the standard reason was the water truck. Someone pointed out that the same story seemed to be told irrespective of which day of the week it was. The class met on Mondays, Wednesdays and Fridays, shifting to Tuesdays, Thursdays and Saturdays some weeks (for a variety of reasons). And yet, the water truck as reason seemed to be uniform, though the trucks did not come every day. This observation led to a very interesting discussion. Water trucks came every third day at that time. Assuming that the trucks came on a Monday, learners realised that within three weeks they would have come on all days of the week.

Speculatively, we asked if the arrival would span all days of the week if the trucks came on alternate days. This was indeed verified to be true. A natural supplementary was to ask whether the observation held for trucks coming every fourth day. At this point, learners had difficulties, so we drew a <u>diagram</u>: the days of the week on a circle, and lines taking us from Monday to Friday to Tuesday, etc. The result was a single closed trajectory that visited all the day-vertices exactly once.

A logical next question was about trucks coming every fifth day (and then, to trucks coming every sixth day), but learners found the question ill motivated and most of them simply refused to "waste time" on these considerations. But then someone pointed out that we could still see what kind of *picture* obtained, whether it was similar to the closed curve we already had for 'every fourth day'. This suggestion met with an enthusiastic response, and the curves were drawn. The conclusion that a "full visit" cycle obtained for "whatever" frequency of truck visit seemed heartening to the learners. I tried to spoil the party with the suggestion that trucks arriving every seventh day would always arrive on one single day and thus the statement was true only for frequency varying from 1 to 6. But this was indignantly dismissed as "obvious and meaningless", since only frequencies from 2 to 6 were "interesting".

The next exercise was to consider a different arrival event, but once every three hours on the clock. My clumsy attempt at story-making met with derision and one of the learners said it was only about drawing pictures, so there was no need for stories! This led to a flurry of drawings, notebooks soon filled with circles and linear trajectories visiting vertices on them. The fact that a frequency of 5 led to a full visit on 12 vertices, but that frequencies of 2, 3, 4 led only to partial visits led to the conjecture that this was about division: if the frequency divided the total, only a partial visit would obtain, but if it did not divide, a full visit was sure. This conjecture was confirmed by 6 and 7 (hailed and celebrated at high decibel levels) but alas, falsified by 8 and 9. Most learners simply gave up and went home at this point.

But then a few persisted, and in a few days' time, not only did we have a rather large collection of drawings (some of them very beautiful), but we also had one of the bright learners identifying the pattern: full visits obtain exactly when the two numbers (frequency and total) were relatively prime (though not stated in this language). I tried to formalize the statement as a theorem, for any k and n, but most learners saw no point in that, seeing it as some mumbo jumbo. The few who were indeed curious that the statement would be true for any k and n, could not see how I could be sure for say 1500 points on a circle, with the curve visiting every 137th successor. I did try to explain that this was possible and that in some sense, that this was what Mathematics was all about, but I did not succeed in the effort.

Within a few years, this activity led to an interesting game with children. Seat n children in a circle, each child numbered *1* to n, remembering her number. A book is passed around, starting with the first child, passing to the k'th neighbour. This is supposed to go on until every child gets the book. Soon children realise that for some values of k and n, everyone gets the book, for some values they don't. Many conjectures are made, and invariably the pattern is discovered. Many pictures are drawn. I have now conducted this activity with many groups of children and teachers, and invariably the moment of discovery comes after these well-defined stages.

However, one thing is clear. In all these discussions, there definitely was argumentation and inference, though it never graduated to proof and universally quantified statements. On the other hand, limited to experimental verification in the small, the learners, be they adults or children, could play around with notions like curves, closed curves and orbits, without ever learning such vocabulary.

Transitions

We often speak of the need to go from the concrete to the abstract in elementary education, especially in the context of Mathematics. But often missed is the realisation that this is a deliberate transition, one that is neither natural nor obvious. A concrete situation or object can be abstracted in many ways, and in the class, we are picking up one particular abstraction (for perhaps very sound reasons). Moreover, after some repetition of such concrete instances abstracted, we want the child to deal with the abstraction per se, leaving behind the concrete realm altogether. This is what I am referring to as a *transition*, moving from one realm to another, often irreversibly.

For instance, when 20 rotis are to be divided equally among 5 persons, it makes sense to act out the division, giving one roti each until all the rotis are exhausted. But when faced with the problem 5624/703, it would be the wrong move to think of distributing 5624 items among 703 persons. Now, why on earth would anyone want to solve such a division problem, at all ? It is highly unlikely that "real life" would ever present us with this problem. The need is entirely mathematical, that of dealing with abstractions like number, division and the patterns visible: 56 / 7 = 8 and 24 / 3 = 8 as well, so one can make a bold guess that the answer is 8, and verify it. Such a facility with abstractions is essential for Mathematics, and students who have not made the transition into this realm, who are yet in the concrete division realm, would find the Mathematics class slipping away from them.

Acknowledging and identifying these transitions is essential for Mathematics curriculum and pedagogy, both at the school and at the college level. Perhaps not surprisingly, these transitions are co-located with what are considered *difficult* topics for teaching/learning. Those who have made the transition need to be engaged in the new realm, those who are yet to make it be given more opportunities. There is no one unique way to make this transition either; recognizing that there are multiple pathways and renewed opportunities is important as well. Understanding these processes also offers hope for solutions to the difficulties mentioned above.

How does one recognize a transition in teaching/learning? Any concept or process that seems difficult to master but seems so obvious and easy once it has been mastered that it is hard to go back to the previous state of learning, involves a transition. This happens when we learn to swim or ride a bicycle. Once you acquire balance, it is almost impossible to return to the wobbly state. Once we learn to factor polynomials, or perform integration, it is impossible to return to the days of early algebra and work out things the way middle school teach us.

This observation lies at the heart of the disconnect between school Mathematics and university Mathematics. A central objective of Mathematics learning is to provide powerful tools that are amazingly general and reliable. When one is equipped with the tool and learns to use it, this renders previously used tools entirely irrelevant. School teaches trigonometry, without which trigonometric functions and calculus cannot be learnt. But having learnt calculus and linear algebra, there is never any need to return to almost any topic taught in school. Later when one learns to use compactness and continuity as a principle, it liberates one from some of the specifics in calculus. Thus the journey continues, and it is one of making many a transition comfortably.

A Map of Transitions

There are many points of transition in Mathematics education, all the way from the primary classroom to the undergraduate class at university. It will be presumptuous on my part to attempt any comprehensive list. Instead let me enumerate some glaring transitions and their pathways.

- Long division: Though multi-digit multiplication involves working out a procedure, it is sufficiently close to the corresponding concrete operation that a transition is not necessitated. Not so in the case of long division; the student deals with an abstract procedure whose correctness or justification becomes clear only after mastering the algorithm itself. But this is possible only if the student can handle the abstractions employed.
- Addition of fractions: When we add natural fractions, we can make stories around them, but when faced with a student who considers 2/3+ 3/5 to be 5/8, the need to make the transition is obvious. The problem

is not lack of understanding LCMs and common denominators, but about *fractions as entities* that we can perform operations like additions on.

- Arithmetic to algebra: This is perhaps the best acknowledged transition in the school curriculum, and algebra is introduced as generalized arithmetic. But even here there are many jumps not negotiated neatly, leading to fall and fracture. For instance, in $x+5=8$, the variable x is a single unique unknown number; in $x+y = 8$ the variable x stands for many unknown numbers (though there are only 9 possibilities if x and y are positive integers); in $x+y = y+x$, the variable x could be any number whatsoever.
- *Additive to multiplicative reasoning*: While it is natural to consider multiplication as repeated addition in the primary school, this makes little sense when faced with *sqrt(2) X sqrt(3).* It is critical to begin seeing multiplication as scaling of some kind. Multiplicative reasoning is crucial to understand the growth of functions, for recognizing similarity in geometric objects and for recognizing and using transformations.
- *From the implicit infinite to the axiomatic infinite*: In school, the infinite is always around, but it is not confronted as such. College Mathematics begins with limits and continuity, by which time infinite objects and sequences are understood in terms of their properties. For instance, consider the question: why does $1/n$ tend to zero as $\underline{n}$ becomes large ?
- *From working with a model to the abstract notion*: The decimal representation of real numbers is known to children in the high school. Unfortunately it gets forgotten that the representation is only a *model*, the notion itself is more general.
- *From the assumed infinite to the explicit finite*: In school, numbers are always around, as big as you want. When one is engaged in combinatorics or number theory problems, one has to work with the explicit finite, and this Is often considered difficuly.
- *Limits and continuity*: Perhaps the biggest experienced discontinuity for students is the epsilon - delta definition of continuity. This is in such an abstract realm, formulated for rigorous foundations, that the demand it makes in terms of a big leap causes many students to be left behind.
- *From inductive to deductive argument*: In middle school, the student is encouraged to observe patterns and generalize them to obtain formulas. Later on the formulas require derivations, proofs. This is due to a cognitive shift that has occurred, whereupon the student is subjected to a standard of proof that is more stringent than what was acceptable earlier.
- *Geometrical reasoning*: Euclidean geometry provides a wonderful opportunity to learn logic in school. The big difficulty in making the transition from factorizing polynomials to such deductions renders many students clueless, they don't know what to look for.
- *Probabilistic reasoning*: This is a unique departure from the rest of Mathematics that involve certainty (theorems). Abstraction and imagination well beyond observable phenomena require comfort with probabilistic reasoning.
- *Mathematical modelling*: This is a part of curricular, but the modelling

never challenges the student's mathematical conceptualization. Indeed, modelling may reintengrate several knowledge domains of Mathematics towards problem solving.

While these can be seen as problem areas that require our attention, there are many simple questions that can pose a big leap. For instance, it seems reasonable that dividing a ribbon of length *m* among three persons should get *m/3* units for each. But what about cutting it up into pieces *3 cm*long generating *m/3* pieces ? Is that possible?

Here are some more questions. Why is *0.99999....= 1* ? What are we assuming here? What is *pi^2,* really speaking ? (Try starting from *pi* as the ratio of circumference to radius of an arbitrary triangle and think of what it means to multiply it by another.)

Bigger than all this is the transition required in a student's predisposition, as she moves from problem solving as the way of obtaining answers to that for gaining insight or constructing arguments. That good problems are those whose solution lead to several new problems, is an essential aspect of doing Mathematics and students who achieve this understanding enter into a new way of thinking altogether.

All this has major implications for the teaching of Mathematics. In Felix Klein's words, Mathematics teachers suffer due to a *double discontinuity*. Many teachers had themselves not negotiated the transitions successfully and lack introspection on these difficulties. When they went from school to college, they moved away from school Mathematics never to return to it for conceptual need. But becoming a teachers requires a backward journey when most of the university Mathematics learnt seems irrelevant. We require knowledgeable teachers, but most teachers do not have personal experience of what it means to do Mathematics over time, exploring questions which have intellectual purpose, not only pedagogic purpose.

This also poses challenges for Mathematics curricula. Allotting equal space for all curricular units is like insisting that everyone should walk at the same speed everywhere. The terrain dictates our ease and speed, and so also is the terrain of mathematical learning: there are easy passes, little streams to jump over, but also brambles to cut through, rocks to climb, and pits to avoid. Once we have a good map on hand and prepare ourselves well, the trek is enjoyable and healthy.